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Proceedings of the 34th
Conference on Decision & Control
New Orleans, LA - December 1995

Planning Motions of Rolling Surfaces*

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Abstract

Rolling between rigid surfaces in space is a well-known nonholonomic system, whose mathematical model has some interesting features that make it a paradigm for the study of some very general systems. It also turns out that the nonholonomic features of this system can be exploited in practical devices with some appeal for engineers. However, in order to achieve all potential benefits, a greater understanding of these rather complex systems and more practical algorithms for planning and controlling their motions are necessary. In this paper, we will consider some geometric and control aspects of the problem of arbitrarily displacing and reorienting a body which rolls without slipping among other bodies.

1 Introduction

Nonholonomic systems have been attracting much attention in the control literature recently, due to both their relevance to practical applications (in particular, to Robotics) and to the challenges that arise in planning and controlling them. Nonholonomic systems commonly encountered in practice can be subdivided in two groups: those where nonholonomy is, so to say, incidental, and basically represents an annoyance for the designer; and those where nonholonomy is introduced on purpose. In the first class one may consider for instance bicycles and cars (possibly with trailers), and space platforms equipped with robotic arms subject to angular momentum conservation. The second group is formed by devices whose nonholonomic behaviour is purposefully introduced and exploited. One of the characteristics of nonholonomic systems that may attract engineers is that in general they can be driven by a small number of inputs (i.e., actuators) with respect to the dimension of their configuration manifold, thus allowing to simplify the hardware design, reducing costs and increasing reliability. Examples of such systems have been reported e.g. by Brockett [1989], Nakamura[1993], Ostrowski *et al.* [1994], Sordalen and Nakamura [1994], Bicchi and Sorrentino [1995].

On the other hand, planning and controlling nonholonomic systems is in general a considerably difficult task. The very fact that fewer degrees-of-freedom are available than configurations involves that standard motion planning techniques can not be directly adapted to nonholonomic systems. From the control viewpoint, nonholonomic systems are intrinsically nonlinear systems, in the sense that they are not exactly feedback linearizable, nor does their linear approximation retain the fundamental characteristics of the system (such as e.g. controllability). Simple (differentiable, time-invariant) feedback control laws, on the other hand, can not be applied to stabilizing nonholonomic systems, as shown by Brockett's theorem [Brockett, 1983].

An important class of nonholonomic systems for which a reasonably satisfactory understanding has been reached in the recent few years is the class of two-inputs nilpotentizable systems that can be put, by feedback transformation, in the so-called "chained" form [Murray and Sastry, 1993]. A complete characterization of such systems (i.e., necessary and sufficient conditions for the existence of a feedback transformation to chained-form) has been provided by Murray [1994], while an algorithm for finding the necessary coordinate transform has been presented by Tilbury, Murray, and Sastry [1995]. As an example, a car pulling an arbitrary number of trailers has been shown to be a chained-form system by Sordalen [1993]. Planning algorithms for chained-form systems in free space have been described by several authors: in his early work Brockett [1981] used sinusoidal inputs, that were subsequently investigated in more detail by Murray and Sastry [1993]. The methods of Lafferriere and Sussmann [1991], Monaco and Normand-Cyrot [1992], and Jacob [1992], using piecewise constant inputs in different arrangements, are particularly well-suited to chained systems, where they achieve exact planning (only approximate, iterative planning schemes are obtained in the general case). Further, chained systems are ***differentially flat*** in the sense of Fliess ***et al.*** [1992], and therefore the techniques of Rouchon ***et al.*** [1993] can be profitably applied. As for the problem of feedback stabilization to a point, time-varying or nonsmooth feedback schemes have been proposed that achieve the goal for chained systems (see for instance [Samson, 1995] and [Sordalen and Egeland, 1995], and references therein).

In this paper, we consider some aspects of the problem of planning nonholonomic systems that can not be put in chained form. In particular, we consider mechanical systems that include bodies rolling on top

*The authors wish to thank Dawn Tilbury for many useful and pleasant discussions. This work was partially supported by the CNR-NSF bilateral project on "Nonlinear Control Methods for Kinematically Defective Non-Holonomic Systems".

0-7803-2685-7/95 $4.00 © 1995 IEEE

of each other without slipping. The fact that such systems are among the simplest (in terms of number of configurations and inputs) exhibiting more general behaviours than chained systems, along with their application potentials in robotics, explains our interest in their investigation.

2 Rolling Motions of Surfaces

2.1 The Plate–Ball Problem

The study of the rolling motion of a sphere on a plane is a classical problem in rational mechanics, recently brought to the attention of the control community by Brockett and Dai [1991].

Consider a ball that rolls without slipping between two horizontal plates, one of which (say the upper) is moved relative to the other. We also assume that friction prevents the ball from spinning about the axis through the contact points. The problem is to move the ball from an initial configuration (position and orientation) to a given final configuration, by means of suitable movements of the upper plate. Among the infinitely many possible solutions to this problem, one may ask to determine the maneuver of the plane that minimizes the length of curve traced out by the sphere on the lower plate. Formally, the problem can be described as an optimal control problem on the five–dimensional Lie group $G = \mathbb{R}^2 \times SO(3)$ of the configurations $g = (x, y, R)$ of the sphere, where $x \in \mathbb{R}^1$, $y \in \mathbb{R}^1$ are the coordinates of the contact point on the lower plate, and $R \in SO(3)$ is a proper 3×3 rotation matrix describing the orientation of a frame fixed with the sphere, with respect to a frame fixed onto the lower plate. The velocity of the sphere is an element of the tangent space at g, $\dot{g} \in T_gG$. Since G is a Lie group (with the group operation $(x_1, y_1, R_1)(x_2, y_2, R_2) = (x_1+x_2, y_1+y_2, R_1R_2)$), we can associate each element of TgG with the Lie algebra of G, $T_eG \cong \mathcal{L}(G) = \mathbb{R}^2 \times so(3)$ (e is the group identity, $e = (0, 0, I)$). We denote by $\mathcal{S}(\mathbf{a}), \mathbf{a} \in \mathbb{R}^3$ a generic element of the Lie algebra $so(3)$ of 3×3 skew–symmetric matrices, with the standard understanding that

$$\mathcal{S}(\mathbf{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix},$$

such that $\mathcal{S}(\mathbf{a})\mathbf{v} = \mathbf{a} \times \mathbf{v}, \forall \mathbf{a}, \mathbf{v} \in \mathbb{R}^3$. Let $V_1 = (v_1, \mathcal{S}(a_1))$ and $V_2 = (v_2, \mathcal{S}(a_2))$ be vector fields in $\mathcal{L}(G)$, then their Lie bracket is defined as $[V_1, V_2] = (0, \mathcal{S}(a_2)\mathcal{S}(a_1) - \mathcal{S}(a_1)\mathcal{S}(a_2))$.

The foregoing optimal control problem can now be written as the minimization of the cost functional

$$L(u_1, u_2) = \int_0^1 \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\, dt, \qquad (1)$$

subject to

$$\begin{aligned} \bar{g}(0) &= g_{start}; & (2)\\ \bar{g}(1) &= g_{goal}; & (3)\\ \dot{g} &= X_1(g)u_1 + X_2(g)u_2, & (4) \end{aligned}$$

where

$$X_1(g) = \begin{bmatrix} 1 \\ 0 \\ -R(g)\mathcal{S}(\mathbf{e}_2) \end{bmatrix}; \; X_2(g) = \begin{bmatrix} 1 \\ 0 \\ R(g)\mathcal{S}(\mathbf{e}_1) \end{bmatrix},$$

and $\mathbf{e}_i$ are the standard unit vectors in $\mathbb{R}^3$. Note that (4) is the conventional form of a linear analytic control system, with inputs u_1 and u_2 representing the components of the velocity of the center of the ball (which is the same as the velocity of the contact point on the fixed plate, and $1/2R$ times the velocity of the moving plate, i.e., the actual physical inputs). Observing that $[-\mathcal{S}(\mathbf{e}_2), \mathcal{S}(\mathbf{e}_1)] = -\mathcal{S}(\mathbf{e}_3)$, by computation of the rank of the control Lie algebra one gets that the system is weakly accessible, and, since no drift term is present, can conclude for the complete controllability of the plate–ball system ([Li and Canny, 1990]; [Jurdjevic, 1993]).

Brockett and Dai [1991] were interested in an approximate version of the problem, which fitted into an ***Engels canonical form*** investigated in that paper. They modified the cost functional to

$$L(u_1, u_2) = \int_0^1 (u_1^2(t) + u_2^2(t))\, dt,$$

which they showed to be equivalent to minimizing arc length for their problem, and provided optimal planning solutions in terms of elliptic integrals (of the first kind). Later on, Jurdjevic [1993] investigated optimal solutions of the complete problem (with cost functional (2.1)), and obtained a full characterization of the solutions, which are also expressed in terms of elliptic integrals (of the third kind). A most interesting aspect of these solutions is that they also minimize the functional

$$\frac{1}{2}\int_0^1 k^2 dt \qquad (5)$$

where $k^2 = \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2}$ is the geodesic curvature of the curve traced by the ball on the fixed plate. In other words, such curve is a solution of the ***elastica*** problem of Euler.

From an engineering viewpoint, the efficiency of computation of a path joining the start and goal configurations is often of more concern than the optimality of its length. This is particularly true when planning has to be executed in the presence of obstacles, in which case a viable solution is to first plan the motion of the object disregarding the nonholonomic constraint, and subsequently approximate such path with a number of nonholonomic paths staying close enough to the holonomic path. To this purpose, the optimal planning methods above described may not be suitable, and more direct procedures have been sought in the literature.

Li and Canny [1990] proposed a planning algorithm based on the use of coordinate–free differential geometric relationships, obtaining an elegant algorithm capable of bringing the sphere to the desired position and orientation by a sequence of three steps. In the first step the sphere center is brought at the desired goal position; in the second step, two orientation parameters of the sphere are settled by executing a closed path of the center, while the third step adjusts the holonomy angle by executing a movement such that the contact point on the sphere follows a latitude circle. While step 1 is straightforward, parameters of motions executed at step 3 are derived from an application of the Gauss–Bonnett theorem, and step 2 is based on an algorithm specific to the geometry of the sphere.

2.1.1 Rolling General Surfaces

To address the more general problem of manipulating an object with general surface by rolling, some tools from the geometry of surfaces are needed. Both the rolling bodies are assumed to be smooth solid surfaces Σ embedded in $\mathbb{R}^3$. The surface of one of the bodies, called the "object", is also assumed to be convex. The other body, whose position is assumed to be fixed in space, will be sometimes referred to as the "finger". We attach to such surfaces local coordinate patches $(\mathbf{f}, U); \mathbf{f} : U \subset \mathbb{R}^2 \rightarrow \Sigma_U \subset \Sigma$, so as to form an atlas. We assume that the coordinate systems are *orthogonal*, i.e. $\mathbf{f}_u^T \mathbf{f}_v = 0$. In these coordinates, a Gauss (normal) map $n : \Sigma \rightarrow S^2 \subset \mathbb{R}^3$, can be written as $n = \frac{\mathbf{f}_u \times \mathbf{f}_v}{\|\mathbf{f}_u \times \mathbf{f}_v\|}$). It is also useful to define a normalized Gauss frame $[x, y, z] = [\mathbf{f}_u/\|\mathbf{f}_u\|, \mathbf{f}_v/\|\mathbf{f}_v\|, n]$, with $\mathbf{f}_u^T \mathbf{f}_v = 0$.

The kinematics of rolling motions can be derived from either the classical differential geometric viewpoint (using first and second fundamental forms for Σ at p, I_p and II_p respectively, and Christoffel symbols of the first and second kind, $[ij, k]$ and Γ_{ij}^k); or using Cartan's definitions of metric form $\mathbf{M}_\Sigma = \mathbf{diag}(\|\mathbf{f}_u\|, \|\mathbf{f}_v\|)$, curvature form $\mathbf{K}_\Sigma = [\mathbf{x}, \mathbf{y}]^T [\mathbf{z}_u, \mathbf{z}_v] \mathbf{M}_\Sigma^{-1}$, and torsion form $\mathbf{T}_\Sigma = \mathbf{y}^T [\mathbf{x}_u, \mathbf{x}_v] \mathbf{M}_\Sigma^{-1}$. While the latter description results more convenient, we recall that the relationship between the two sets of forms is given by $\mathbf{M}_\Sigma = \sqrt{I_p}$, $\mathbf{K}_\Sigma = \mathbf{M}_\Sigma^{-T} II_p \mathbf{M}_\Sigma^{-1}$, and $\mathbf{T}_\Sigma \mathbf{M}_\Sigma = \mathbf{M}_{22} \mathbf{M}_{11}^{-1} [\Gamma_{11}^2, \Gamma_{12}^2]$. (cf. e.g. Murray, Li and Sastry[1994]).

The kinematic equations of motion of the contact points between two bodies rolling on top of each other describe the evolution of the (local) coordinates of the contact point on the finger surface, $\alpha_f \in \mathbb{R}^2$, and on the object surface, $\alpha_o \in \mathbb{R}^2$, along with the (holonomy) angle between the x–axes of the two gauss frames ψ, as they change according to the rigid relative motion of the finger and the object described by the relative velocity $\mathbf{v}$ and angular velocity ω. According to the derivation of Montana [1988], in the presence of friction one has

$$\begin{aligned} \dot{\alpha}_f &= \mathbf{M}_f^{-1} \mathbf{K}_r^{-1} \begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix}; \\ \dot{\alpha}_o &= \mathbf{M}_o^{-1} \mathbf{R}_\psi \mathbf{K}_r^{-1} \begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix}; \\ \dot{\psi} &= \mathbf{T}_f \mathbf{M}_f \dot{\alpha}_f + \mathbf{T}_o \mathbf{M}_o \dot{\alpha}_o; \end{aligned} \qquad (6)$$

where $\mathbf{K}_r = \mathbf{K}_f + \mathbf{R}_\psi \mathbf{K}_o \mathbf{R}_\psi$ is the relative curvature form, and

$$\mathbf{R}_\psi = \begin{bmatrix} \cos\psi & -\sin\psi \\ -\sin\psi & -\cos\psi \end{bmatrix}.$$

We note explicitly that, while this formulation of the kinematics of rolling motions differs from the one used earlier in this paper, simple geometrical relationships exist relating the representation of the object orientation by means of $R \in SO(3)$ and that employing the contact point coordinates α_o, α_f and the holonomy angle ψ.

The rolling kinematics (6) are readily written in the standard control form, $\dot{\xi} = \mathbf{g}_1(\xi) v_1 + \mathbf{g}_2(\xi) v_2$, if we consider a local parametrization of the configuration manifold as given by the state vector $\xi \in \mathbb{R}^5, \xi = [x, y, u, v, \psi]^T$ and take the angular velocities of the rolling object as the system inputs, $v_1 = \omega_x$ and $v_2 = \omega_y$. In the case of a unit sphere rolling on a plane, for instance, the control vector fields are

$$\mathbf{g}_1(\xi) = \begin{bmatrix} 0 \\ -1 \\ \frac{S_\psi}{C_v} \\ C_\psi \\ T_v S_\psi \end{bmatrix}; \quad \mathbf{g}_2(\xi) = \begin{bmatrix} 1 \\ 0 \\ \frac{C_\psi}{C_v} \\ -S_\psi \\ T_v C_\psi \end{bmatrix},$$

where the shorthand notation $S_\alpha, C_\alpha, T_\alpha$ for $\sin(\alpha), \cos(\alpha)$ and $\tan(\alpha)$, respectively, is used. By computing the controllability filtration

$$\begin{aligned} \Gamma_0 &= \text{span}\,(\mathbf{g}_1, \mathbf{g}_2) \\ \Gamma_1 &= \Gamma_0 + [\Gamma_0, \Gamma_0] \\ \Gamma_2 &= \Gamma_1 + [\Gamma_1, \Gamma_0] \\ &\vdots \\ \Gamma_k &= \Gamma_{k-1} + [\Gamma_{k-1}, \Gamma_0], \end{aligned}$$

and its associated growth vector,

$$\gamma = [\dim \Gamma_1,\ \dim \Gamma_2,\ \ldots,\ \dim \Gamma_k],$$

one obtains $\gamma = [2,\ 3,\ 5,\ 5,\ \ldots]$ at every ξ except where the parametrization of the configuration manifold is singular. In their controllability argument, Li and Canny [1990] circumvented the latter problem by using a different chart of the atlas covering the sphere.

Our interest here is however in pointing out that (even in this simplest plate–ball example), nonholonomic system comprised of rolling surfaces do not fit conditions for most known exact planning methods to be applied. In fact, since $\dim \Gamma_i \neq i + 2$, there does not exist any state and feedback transforms that can put the system in chained form (Murray, 1994]. Similarly, Rouchon *et al.* [1993] observed that the system is not differentially flat. On the other hand, system (6) is not in nilpotent form, so that application of the constructive method of Lafferriere and Sussmann [1991] would only provide approximate results. Furthermore, direct application of multirate digital control techniques to the system (6) is not possible, since the corresponding exact sampled model is not available.

Noting that any system with $n \leq 4$ states and $m = 2$ inputs can be put in chained form ([Hermes, 1989]), and hence is differentially flat, nilpotent, and its sampled model can be exactly computed, it can be observed that rolling systems with $n = 5, m = 2$ are in a sense the simplest systems to which powerful known methods fail to apply.

We return to the case of an object with general smooth, convex surface rolling on top of a plane.

Notwithstanding the genericity of its Lie filtration growth, the rolling kinematic equations do possess a structure that can be exploited to find efficient planning algorithms. An useful result in this sense is the following

Proposition 1 ***There exist a state diffeomorphism and a regular static state feedback law such that the kinematic equations of contact (6) for planar fingers assume a strictly triangular structure (as defined e.g. in [Murray and Sastry, 1993]).***

Proof. Rewrite (6) as

$$\begin{aligned} \dot{\alpha}_f &= \mathbf{M}_f^{-1}\mathbf{K}_r^{-1}\mathbf{w}; \\ \dot{\alpha}_o &= \mathbf{M}_o^{-1}\mathbf{R}_\psi\mathbf{K}_r^{-1}\mathbf{w}; \\ \dot{\psi} &= [\mathbf{T}_f\mathbf{R}_\psi + \mathbf{T}_o]\mathbf{K}_r^{-1}\mathbf{w}, \end{aligned} \tag{7}$$

where $\mathbf{w}^T = [-\omega_y \; \omega_x]$. Recall that for plane fingers, $\mathbf{T}_f = [0 \; 0]$, and $\mathbf{M}_f = \mathbf{I}_2$. Define the regular state feedback $\mathbf{w} = \gamma(\alpha_f, \alpha_o, \psi)\hat{\mathbf{w}}$ as

$$\gamma(\alpha_f, \alpha_o, \psi) = \mathbf{K}_r\mathbf{M}_o\hat{\mathbf{w}}, \tag{8}$$

and apply a change of coordinates that suitably re-orders the states, to obtain

$$\begin{aligned} \dot{\alpha}_o &= \hat{\mathbf{w}}; \\ \dot{\psi} &= \mathbf{T}_o\mathbf{M}_o\hat{\mathbf{w}}; \\ \dot{\alpha}_f &= \mathbf{R}_\psi\mathbf{M}_o\hat{\mathbf{w}}, \end{aligned}$$

which is strictly lower triangular. □

As an instance of application of this technique, consider the case of an object with an axial symmetry rolling on a planar finger. Axial-symmetric objects are convenient for computations, since a single patch of cylindrical coordinates provides an orthogonal parametrization of the whole surface except at the north and south poles, and at one meridian. Let such coordinate system be $(\mathbf{f}, (-\pi, \pi) \times \mathbb{R})$,

$$\mathbf{f} = \begin{bmatrix} \rho(u,v)C_v \\ \rho(u,v)S_u \\ v \end{bmatrix},$$

and notice that, for systems with an axial symmetry, $\frac{\partial \rho}{\partial u} = 0$. Denoting $\frac{d\rho}{dv} = \rho_v$, evaluating the surface forms and applying the triangularizing feedback above, the control system associated with the rolling kinematic equations is obtained as $\dot{\mathbf{q}} = \mathbf{g}_1(\mathbf{q})v_1 + \mathbf{g}_2(\mathbf{q})v_2$, with $\mathbf{q} = [u, v, \psi, x, y]^T$ and

$$\mathbf{g}_1(\mathbf{q}) = \begin{bmatrix} 1 \\ 0 \\ -\frac{\rho_v}{\sqrt{1+\rho_v{}^2}} \\ C_\psi \rho \\ -S_\psi \rho \end{bmatrix}; \; \mathbf{g}_2(\mathbf{q}) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\sqrt{1+\rho_v{}^2}S_\psi \\ -\sqrt{1+\rho_v{}^2}C_\psi \end{bmatrix}.$$

Objects with an axis of symmetry are of practical interest in industrial parts handling applications, for instance. Bicchi and Sorrentino [1995] discussed the design of a dextrous robot hand for manipulating objects by rolling them between two plane fingers (see

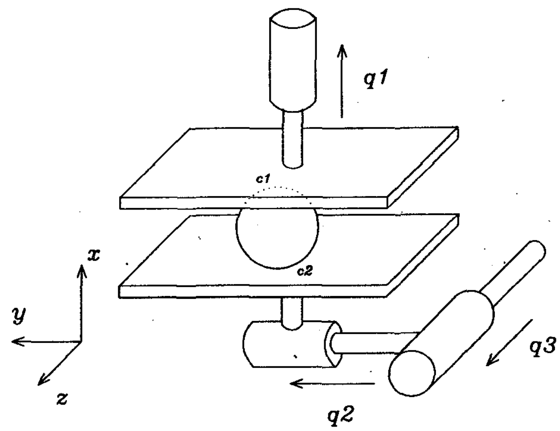

Figure 1: The nonholonomic dextrous hand developed at the University of Pisa

fig. 1). Exploitation of nonholonomy allowed the hand to be built using only three actuators, with substantial savings in terms of cost, weight, and failure likelyhood with respect to other dextrous hands using actuators in numbers ranging between 10 and 30. The viability of such a solution is subject to the validity of the conjecture that controllability of rolling motions between surfaces is generic, i.e., almost every pair of surfaces form a controllable rolling kinematic system. An argument in favour of such conjecture is that Li and Canny [1990] showed that in the rolling of a sphere on top of another sphere, controllability is lost only when the radii are coincident or when either vanishes. Based on the developments above, we can give here another partial argument in support of that conjecture:

Proposition 2 ***The kinematic system comprised of any smooth strictly convex axial-symmetric surface rolling on a plane is controllable.***

Proof. The Lie brackets of the control vector fields are computed as

$$\mathbf{g}_3 = [\mathbf{g}_1, \mathbf{g}_2] = \begin{bmatrix} 0 \\ 0 \\ \frac{\rho_{vv}}{(1+\rho_v^2)^{3/2}} \\ 0 \\ 0 \end{bmatrix};$$

$$\mathbf{g}_4 = [\mathbf{g}_1, \mathbf{g}_3] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\rho_{vv}}{(1+\rho_v^2)^{3/2}}\rho S_\psi \\ \frac{\rho_{vv}}{(1+\rho_v^2)^{3/2}}\rho C_\psi \end{bmatrix}$$

$$\mathbf{g}_5 = [\mathbf{g}_2, \mathbf{g}_3] = \begin{bmatrix} 0 \\ 0 \\ \frac{\rho_{vvv}(1+\rho_v^2) - 3\rho_v\rho_{vv}^2}{(1+\rho_v^2)^{5/2}} \\ \frac{\rho_{vv}}{(1+\rho_v^2)}C_\psi \\ -\frac{\rho_{vv}}{(1+\rho_v^2)}S_\psi \end{bmatrix}$$

The growth vector of the controllability filtration is [2, 3, 5, 5, ...] whenever the distribution $span\{g_1\ g_2\ g_3\ g_4\ g_5\}$ is full rank. The singularities of the distribution are at the roots of either of the equations

$$\rho = 0; \tag{9}$$

$$\rho_{vv} = 0; \tag{10}$$

Condition (9) indicates that the distribution is singular when the object degenerates to a point (infinite curvature surface). Note that, for convex surfaces with finite curvature, the radius ρ can only vanish at the north and south poles (no hourglasses allowed). The poles are not covered by the above described cylindrical coordinate patch anyhow.

Condition (10) corresponds to surfaces which are not strictly convex. In fact, the curvature form for surfaces of revolution in cylindrical coordinates is evaluated as

$$\mathbf{K}_o = \begin{bmatrix} \frac{1}{\rho\sqrt{1+\rho_v^2}} & 0 \\ 0 & \frac{-\rho_{vv}}{(1+\rho_v^2)^{3/2}} \end{bmatrix}.$$

Note incidentally that such surfaces may still be controllable, with a higher local degree of nonholonomy. Surfaces with $\rho_{vv}(v) \equiv 0$ are cones and cylinders with linear generators. For such surfaces, the growth vector is [2, 0, 0, ...], hence cones and cylinders (as well as the point surface) are actually noncontrollable.

The proof of global controllability for convex surfaces can be finalized by defining other suitable coordinate patches to cover the borders of the cylindrical patch (the meridian $u = -\pi$ and the north and south poles of the object), and going again through the Lie algebra rank condition calculations. □

3 Applications to Planning

The relevance of the strictly triangular form above derived to planning is in the relative ease by which the flows of the vectorfields can be integrated (the term "integration" for "solution" of an ODE is used properly in this case). In the plate–ball example, for instance, the state feedback law

$$\mathbf{w} = \begin{bmatrix} C_v S_\psi & C_\psi \\ C_v C_\psi & -S_\psi \end{bmatrix} \hat{\mathbf{w}} \tag{11}$$

transforms (6) in

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{\psi} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ S_v \\ C_\psi C_v \\ -S_\psi C_v \end{bmatrix} \hat{w}_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -S_\psi \\ -C_\psi \end{bmatrix} \hat{w}_2. \tag{12}$$

One has therefore, for any constant $\delta \in \mathbb{R}$

$$\Phi_t^{\delta g_1} = \begin{bmatrix} u_0 + \delta t \\ v_0 \\ \psi_0 + \delta t S_{v_0} \\ x_0 + \frac{1}{T_{v_0}} [\sin(\psi_0 + \delta t S_{v_0}) - S_{\psi_0}] \\ y_0 + \frac{1}{T_{v_0}} [\cos(\psi_0 + \delta t S_{v_0}) - C_{\psi_0}] \end{bmatrix}, \tag{13}$$

and

$$\Phi_t^{g_2} = \begin{bmatrix} u_0 \\ v_0 + \delta t \\ \psi_0 \\ x_0 - \delta t S_{\psi_0} \\ y_0 - \delta t C_{\psi_0} \end{bmatrix}. \tag{14}$$

A solution to the planning problem for system (6) can now be applied, consisting in a particular arrangement of piecewise constant inputs. In fact, by concatenating a sequence of constant inputs of the form

$$\begin{cases} \hat{w}_1 = \bar{w}_{1,1} \\ \hat{w}_2 = 0 \end{cases}, \quad 0 < t < T$$

$$\begin{cases} \hat{w}_1 = 0 \\ \hat{w}_2 = \bar{w}_{2,1} \end{cases}, \quad T < t < 2T$$

$$\vdots$$

$$\begin{cases} \hat{w}_1 = \bar{w}_{1,k} \\ \hat{w}_2 = 0 \end{cases}, \quad 2kT < t < (2k+1)T$$

The 2k+1 unknown variables $\bar{w}_{1,i}, \bar{w}_{2,i}$ can be evaluated by solving the system of five nonlinear equations obtained by equating the final to the desired configuration, namely

$$\Phi_{T_{2k+1}-T_{2k}}^{\bar{w}_{1,k}g_1} \circ \ldots \circ \Phi_{T_2-T_1}^{\bar{w}_{2,1}g_2} \circ \Phi_{T_1}^{\bar{w}_{1,1}g_1}(\mathbf{x}_0) - \mathbf{x}_{des} = 0 \tag{15}$$

Naturally, other concerns such as minimizing the length of the path or avoiding limits of the workspace can be taken into account by building a suitable optimization problem constrained by (15). In [Bicchi and Sorrentino, 1995] are reported the results of the application of this method to planning for the plate–ball problem.

4 Discussion

While the fact that the alternating control scheme introduced above works for planning local motions descend directly from the controllability of the system (the arguments in the proof of controllability for nonlinear systems rely precisely on such a construction of the control sequence, see e.g. [Hermann and Krener, 1977]), what is the minimum number of control steps that guarantees the existence of a solution for the generic motion in the large (in particular, whether such number is $2k+1 = n = 5$), is an open problem. A close relationship with other piecewise constant input based methods is observed, in particular with the multirate schemes of Monaco and Normand–Cyrot [1992].

The planning technique based on the triangularized form above introduced can be applied to more general cases, including that of a general surface of revolution between flat fingers. For even more general surfaces, a continuation algorithm of Sussman [1993] can be applied. In its practical implementation this method, just like other related approximate iterative techniques suffers from an excessive demand of time for planning. Also, the failure of the method because of abnormal extremals encountered along the path to be lifted , is possible theoretically. Research in this direction is also being undertaken.

One of the main problems in the actual implementation of the technique on manipulation systems is that, due to the feedback transformation used, control inputs used for planning are not the available physical inputs (say joint velocities in a robot hand), but rather complex functions of system states evolving along the planned trajectory. Furthermore, in many practical implementations it is hardly reasonable to expect that the full state vector is available for measurements. In particular, object orientation angles are difficult to measure . While it is possible to integrate the kinematic equations for the sphere to obtain desired joint trajectory, this is difficult for objects of general shape. Moreover, such approach would result in a completely open-loop control scheme. In [Bicchi and Sorrentino, 1995] a technique based on controlling the coordinates of the contact point on the lower finger so as to track the trajectory resulting from planning, by using real-time tactile feedback, is described and experimental results are reported.

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