43rd IEEE Conference on Decision and Control
December 14-17, 2004
Atlantis, Paradise Island, Bahamas

A Straightforward Approach to the Cheap LQ Problem for Continuous-Time Systems in Geometric Terms

Domenico Prattichizzo Giovanni Marro Lorenzo Ntogramatzidis

Abstract— **This paper addresses the cheap version of the classical linear quadratic (LQ) optimal control problem for continuous-time systems. The approach herein considered differs from those presented in literature, since it consists of applying the tools of the geometric control theory to the Hamiltonian system. In this way, it is possible to compute the stabilizing state-feedback gain achieving optimality by using standard geometric algorithms, whenever the initial state satisfies a suitable necessary and sufficient condition for solvability, also stated in geometric terms.**

I. INTRODUCTION

The infinite-horizon linear quadratic (LQ) regulator is a well-known and deeply investigated problem in control theory. When the matrix weighting the input function in the quadratic cost, traditionally denoted by R, is positive definite, the problem is said to be *regular*. It is well-known that if the system is stabilizable and the Hamiltonian matrix has no purely imaginary eigenvalues, a stabilizing state-feedback control minimizing the quadratic cost exists for any arbitrarily assigned initial state. The corresponding optimal gain matrix is expressed as a function of the stabilizing solution of a suitable algebraic Riccati equation (see e.g. [1], [8]). If R is positive semidefinite, the problem is called *singular*, or *cheap* in the particular case when R is zero. This subject has been dealt with by using the theory of the geometric approach (see for example [6], [17], [12] and references therein). In particular, in [6] and [17] it has been clearly shown that an optimal solution to the singular problem always exists if the class of allowable controls is extended to include distributions, hence not obtainable by an algebraic state-feedback. To this purpose, the relevant concepts of weak unobservability and strong reachability have been introduced. A geometric algorithm enables singular problems to be reformulated in a regular form, leading to reduced order algebraic Riccati equations. The approach presented in [12] is based on a detailed analysis of the structure of both singular and cheap problems (where the latter is treated as the limiting case of the former), by the exploitation of the so-called *special coordinate basis*. Hence, a complete characterization of the finite-infinite zero structure, as well as the input-output decoupling zeros, is provided. Valuable results on this subject have been also presented in [13], [15], [14] and references therein, which are based on linear matrix inequalities. The latter provide a unified framework for the solution of both regular and singular problems.

Differently from the above-mentioned contributions on this topic, the geometric approach is here exploited to provide a detailed insight into the structure of the Hamiltonian system, which, along with the correct boundary conditions, represents a set of necessary and sufficient conditions for optimality. In particular, it is shown that an optimal solution exists if and only if the initial condition belongs to the projection of a suitable internally stabilizable controlled invariant subspace of the Hamiltonian system onto the state-space of the original system. If this condition is met, the optimal gain can be easily determined by using the standard algorithms of the geometric approach applied to the Hamiltonian system, by simply recasting the cheap control problem as a perfect decoupling problem.

This work is intended to be the counterpart of [9] for continuous-time systems. However, many structural differences can be found between the properties of the Hamiltonian system in the continuous and discrete time case. Moreover, in the continuous time domain an optimal state-feedback does not exist for any arbitrarily assigned initial condition. Hence, the problem of finding the subspace of initial conditions for which the problem admits a solution is relevant, and will be solved geometrically.

Notation. Throughout this paper, the symbol $\mathbb{R}^{n\times m}$ will denote the space of $n\times m$ real matrices. Matrices and linear maps are denoted by slanted capitals, whereas vector spaces are denoted by script capitals. The image and the null-space of matrix $M\in\mathbb{R}^{n\times m}$ are respectively denoted by $\operatorname{im} M$ and $\ker M$. Given a subspace $\mathcal{Y}$ of $\mathbb{R}^n$, the symbol $M^{-1}\mathcal{Y}$ stands for the inverse image of $\mathcal{Y}$ with respect to the linear transformation M, i.e., it is the locus of all vectors of $\mathbb{R}^m$ that are mapped by M into $\mathcal{Y}$. Denote by M^T and by M^+ the transpose and the Moore-Penrose pseudo-inverse of M, respectively; since the identity $(M^{-1})^T=(M^T)^{-1}$ holds in general, the symbol M^{-T} will concisely be used. The origin of the vector space $\mathbb{R}^n$ will be referred to by the symbol 0_n, whereas the symbol I_n will stand for the $n\times n$ identity matrix. Finally, the symbol $\mathbb{R}_+$ will denote the set of non-negative real numbers.

D. Prattichizzo is with Dipartimento di Ingegneria dell' Informazione, Università di Siena, via Roma, 77 – 53100 Siena, Italy prattichizzo@dii.unisi.it

G. Marro, L. Ntogramatzidis are with Dipartimento di Elettronica, Informatica e Sistemistica, Università di Bologna, viale Risorgimento, 2 – 40136 Bologna, Italy {gmarro, lntogramatzidis}@deis.unibo.it

II. DESCRIPTION OF THE PROBLEM

Consider the linear time-invariant system modeled by the equations

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \qquad x(0) = x_0 \in \mathbb{R}^n \\ y(t) &= C\,x(t) \end{aligned} \tag{1}$$

where, for all $t \geq 0$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input and $y(t) \in \mathbb{R}^p$ is the output, whereas A, B and C are real constant matrices of proper sizes. Furthermore, no generality is lost if we assume that B has linearly independent columns and C has linearly independent rows.

System (1) will be referred to as the triple (A, B, C) and will be concisely denoted by the symbol Σ.

The geometric approach setting will require the following notations: $\mathcal{V}_\Sigma^*$ stands for the maximum $(A, \operatorname{im} B)$-controlled invariant subspace of the triple (A, B, C) contained in the null-space of C, which is also referred to as $\max \mathcal{V}(A, \operatorname{im} B, \ker C)$. The symbol $\mathcal{S}_\Sigma^*$ stands for the minimum $(A, \ker C)$-conditioned invariant subspace containing the image of B, which is also referred to as $\min \mathcal{S}(A, \ker C, \operatorname{im} B)$.

Problem 1. Refer to system (1). Assume that

(A1) the pair (A, B) is stabilizable

(A2) Σ has no invariant zeros on the imaginary axis

Determine the subspace of initial conditions yielding optimal regulation through a stabilizing algebraic state-feedback control $u(t) = -K\,x(t)$, i.e., such that

1. the closed-loop matrix $A - B\,K$ is stable
2. the corresponding state trajectory minimizes the quadratic performance index

$$J(x, u) := \frac{1}{2}\int_0^\infty y^T(t)\,y(t)dt = \frac{1}{2}\int_0^\infty x^T(t)C^T C x(t)dt$$

If such condition is met, find the state-feedback matrix K achieving optimality.

III. A GEOMETRIC INSIGHT INTO THE STRUCTURE OF THE HAMILTONIAN SYSTEM

For the reader's convenience, the following standard properties of the geometric approach theory are briefly recalled.

Property 1: The following statements are equivalent:

1) Σ is left invertible
2) $\mathcal{V}_\Sigma^* \cap \mathcal{S}_\Sigma^* = 0_n$
3) $B^{-1}\,\mathcal{V}_\Sigma^* = 0_m$

(see [4, p.237] and [16, p.189]). Roughly speaking, if Σ is not left invertible, the input function corresponding to a given response (obtained with zero initial condition) can only be determined modulo $B^{-1}\,\mathcal{V}_\Sigma^* \subseteq \mathbb{R}^m$ (see [3, Theorem 3]. Thus, the subspace $B^{-1}\,\mathcal{V}_\Sigma^*$ is also called *input unobservability subspace*.

Property 2: The following statements are equivalent:

1) Σ is right invertible
2) $\mathcal{V}_\Sigma^* + \mathcal{S}_\Sigma^* = \mathbb{R}^n$
3) $C\,\mathcal{S}_\Sigma^* = \mathbb{R}^p$

If Σ is not right invertible, the output function can be imposed modulo any complement of the subspace $C\,\mathcal{S}_\Sigma^* \subseteq \mathbb{R}^p$ (see [3, Theorem 4]. Thus, the subspace $C\,\mathcal{S}_\Sigma^*$ is called *output reachability subspace*.

Denote by Σ^T the adjoint of system Σ, described by the triple (A^T, C^T, B^T).

Property 3: System Σ is left invertible (resp. right invertible) if and only if Σ^T is right invertible (resp. left invertible).

The proof of the former follows from the well-known identities of the geometric approach (see [4, p.209]):

$$\mathcal{V}_\Sigma^* = (\mathcal{S}_{\Sigma^T}^*)^\perp \tag{2}$$

$$\mathcal{S}_\Sigma^* = (\mathcal{V}_{\Sigma^T}^*)^\perp \tag{3}$$

Recall that, given a friend F of $\mathcal{V}_\Sigma^*$, i.e. a matrix such that $(A + B\,F)\,\mathcal{V}_\Sigma^* \subseteq \mathcal{V}_\Sigma^*$, the eigenvalues of $A + B\,F$ restricted to $\mathcal{V}_\Sigma^*$ are split into two sets. The eigenvalues of $A + B\,F$ which are internal to $\mathcal{V}_\Sigma^* \cap \mathcal{S}_\Sigma^*$ depend on the choice of F, in the sense that they are all freely assignable by a suitable choice of F. The eigenvalues of $A + B\,F$ restricted to $\mathcal{V}_\Sigma^*$ which are external to $\mathcal{V}_\Sigma^* \cap \mathcal{S}_\Sigma^*$ are fixed for any friend of $\mathcal{V}_\Sigma^*$, and are called *invariant zeros* of Σ. In symbols

$$\mathcal{Z}(\Sigma) = \sigma(A + BF)_{\mathcal{V}_\Sigma^*/\mathcal{V}_\Sigma^* \cap \mathcal{S}_\Sigma^*}$$

If all the invariant zeros of Σ have strictly negative real part, Σ is said to be *minimum phase* and $\mathcal{V}_\Sigma^*$ is said to be *internally stabilizable*. The subspace $\mathcal{V}_\Sigma^* \cap \mathcal{S}_\Sigma^*$ can be interpreted as the locus of states that can be reached from the origin with state trajectories all contained in $\mathcal{V}_\Sigma^*$, hence it is often referred to as the *reachable subspace* on $\mathcal{V}_\Sigma^*$ (see [10]). For a detailed analysis of the internal and external eigenstructure of a controlled invariant subspace see [4, pp.217-222] and [16, pp.89-96].

By virtue of Property 1, if Σ is left invertible, the set of invariant zeros reduces to the set of eigenvalues of $A + B\,F$ restricted to $\mathcal{V}_\Sigma^*$.

The time-reversed representation of system (1) can be defined as follows: the state dynamics can be written backwards in time as

$$-\dot{x}(t) = -A\,x(t) - B\,u(t) \tag{4}$$

while the output equation does not change. The triple $(-A, -B, C)$, herein denoted by Σ^{-1}, will be referred to as the time-reversed system associated with Σ. Note that $\mathcal{V}_\Sigma^* = \mathcal{V}_{\Sigma^{-1}}^*$ and $\mathcal{S}_\Sigma^* = \mathcal{S}_{\Sigma^{-1}}^*$, since the definitions of $\mathcal{V}_\Sigma^*$ and $\mathcal{S}_\Sigma^*$ do not depend on the sign of the matrices A and B. Hence, the following property holds.

Property 4: System Σ is left invertible (resp. right invertible) if and only if its time-reversed representation Σ^{-1} is such.

Recall that the optimal state trajectory and control law for Problem 1 satisfy the Hamiltonian system, obtained by extending the state $x(t)$ of system (1) with the co-state function $\lambda(t) \in \mathbb{R}^n$ $(t \geq 0)$, which can be written as

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \\ \hat{y}(t) &= \begin{bmatrix} 0 & B^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = 0 \end{aligned}$$

and is obtained by the computation of the derivatives of the Hamiltonian function with respect to $x(t)$, $\lambda(t)$ and $u(t)$ (see to this purpose [8, pp.131-133]). The matrices in the former will be respectively denoted by the symbols $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$, while the triple $(\widehat{A}, \widehat{B}, \widehat{C})$ will be denoted by $\widehat{\Sigma}$. The structure of the Hamiltonian system, which along with the boundary equations represents a set of necessary and sufficient conditions for optimality, points out that the problem is that of finding the control law that mantains $\hat{y}(t) = 0$ for all $t \geq 0$ for the assigned initial condition $x(0) = x_0$, and such that the corresponding state trajectory converges to the origin as t approaches infinity. Clearly, this aim can be achieved if and only if the initial condition x_0 is such that an initial value of the co-state $\lambda(0) := \lambda_0$ exists so as that $\begin{bmatrix} x_0^T & \lambda_0^T \end{bmatrix}^T$ belongs to an internally stabilizable $(\widehat{A}, \operatorname{im} \widehat{B})$-controlled invariant subspace contained in the null-space of $\widehat{C}$. Note that this means exactly that the initial state x_0 belongs to the projection of this $(\widehat{A}, \operatorname{im} \widehat{B})$-controlled invariant subspace in $\mathbb{R}^{2n}$ on the state-space of Σ.

The following lemma provides an important characterization of the Hamiltonian system in terms of left and right invertibility. The further hypotheses of left invertibility of Σ is technical in nature and will be removed in Section V.

Lemma 1: Let Σ be left invertible. Then, the Hamiltonian system $\widehat{\Sigma}$ is both left and right invertible.

Proof: First, note that $\widehat{\Sigma}$ is the series connection of Σ and the time-reversed representation of the adjoint of Σ, henceforth denoted by Σ^{-T}, and representing the triple $(-A^T, -C^T, B^T)$. Consider Figure 1. As a consequence of

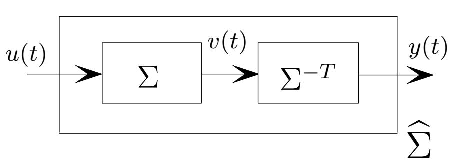

Fig. 1. Inner structure of the Hamiltonian system

Properties 3 and 4, it follows that Σ^{-T} is right invertible. Now we prove that the overall system $\widehat{\Sigma}$ is right invertible. Suppose that the initial states of Σ and Σ^{-T} are zero. Since Σ^{-T} is right invertible but, in general, not left invertible, a one-to-one map exists between the input functions on the orthogonal complement of its input unobservability subspace $(C^T V^*_{\Sigma^T})^\perp$ and its output functions. On the other hand, since Σ is left invertible but, in general, not right invertible, a one-to-one map exists between its input functions and what is obtained on its output reachability subspace $C^T S^*_{\Sigma}$, which is exactly[1] the orthogonal complement of the input unobservability subspace of Σ^{-T}. As a result, a one-to-one map exists between the input functions of Σ and the output functions of Σ^{-T}. Hence, the overall system $\widehat{\Sigma}$ is right invertible. Finally, notice that the time-reversed representation of the Hamiltonian system $\widehat{\Sigma}^{-1}$ and its adjoint $\widehat{\Sigma}^T$ are linearly equivalent, i.e., the triple $(-\widehat{A}, -\widehat{B}, \widehat{C})$ is obtained by the triple $(\widehat{A}^T, \widehat{C}^T, \widehat{B}^T)$ through the coordinate transformation in $\mathbb{R}^{2n}$ given by

$$T = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

As a result, $\widehat{\Sigma}^{-1}$ and $\widehat{\Sigma}$ are left invertible, since $\widehat{\Sigma}^T$ is such. ■

In the sequel, the symbols $\mathcal{V}^*_{\widehat{\Sigma}} := \max \mathcal{V}\,(\widehat{A}, \operatorname{im} \widehat{B}, \ker \widehat{C})$ and $\mathcal{S}^*_{\widehat{\Sigma}} := \min \mathcal{S}\,(\widehat{A}, \ker \widehat{C}, \operatorname{im} \widehat{B})$ will be used, consistently with the notation previously introduced.

Corollary 1: Let Σ be left invertible. The set of invariant zeros of the Hamiltonian system $\widehat{Sigma}$ consists of all the invariant zeros of the system Σ along with their opposite, hence they are all coupled by pairs of the type $(z, -z)$.

Proof: Let z be an invariant zero of the Hamiltonian system $\widehat{\Sigma}$, i.e., it is an eigenvalue of $(\widehat{A} + \widehat{B}\,\widehat{F})$ restricted to $\mathcal{V}^*_{\widehat{\Sigma}}$, where $\widehat{F}$ is a friend of $\mathcal{V}^*_{\widehat{\Sigma}}$, since, owing to Lemma 1, $\widehat{\Sigma}$ is left invertible. It follows that z is an invariant zero of $\widehat{\Sigma}^T$, hence it is also an invariant zero of $\widehat{\Sigma}^{-1}$, since in Lemma 1 it has been shown that $\widehat{\Sigma}^T$ and $\widehat{\Sigma}^{-1}$ are linearly equivalent. However, since $\mathcal{V}^*_{\widehat{\Sigma}} = \mathcal{V}^*_{\widehat{\Sigma}^{-1}}$ and the eigenvalues of $(\widehat{A} + \widehat{B}\,\widehat{F})$ and those of $(-\widehat{A} - \widehat{B}\,\widehat{F})$ are opposite, it is found that $-z$ is an invariant zero of $\widehat{\Sigma}$. ■

The following Lemma holds.

Lemma 2: Let Σ be left invertible, and let r be the dimension of $\mathcal{S}^*_{\Sigma}$. Hence, the following equalities hold:

$$\dim \mathcal{S}^*_{\widehat{\Sigma}} \;=\; 2\,r \tag{5}$$

$$\dim \mathcal{V}^*_{\widehat{\Sigma}} \;=\; 2\,(n-r) \tag{6}$$

Proof: Consider Figure 2. Let U be a basis matrix of

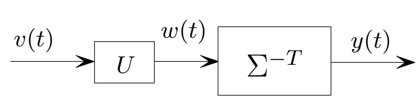

Fig. 2. Structure of the modified system $\bar{\Sigma}$

[1]For computational purposes, recall that if $M \in \mathbb{R}^{n \times m}$ is a linear map and $\mathcal{Y}$ is a subspace of $\mathbb{R}^n$, it is found that ([4, Property 3.1-3, p.128])

$$M^T\,\mathcal{Y}^\perp = (M^{-1}\,\mathcal{Y})^\perp$$

the subspace $C\,\mathcal{S}^*_\Sigma$ of $\mathbb{R}^p$, and denote by the symbol $\bar{\Sigma}$ a new system referred to the triple $(A^T, C^T U, B^T)$, which is right and left invertible. The right invertibility directly follows from the arguments used in the proof of Lemma 1 concerning the bijection existing between the output functions of Σ^{-T} and the projection of functions $v(\cdot)$ on the subspace $C\,\mathcal{S}^*_\Sigma$. Indeed, since U is a basis matrix of $C\,\mathcal{S}^*_\Sigma$, an arbitrary input function $w(\cdot)$ on $C\,\mathcal{S}^*_\Sigma$ can be obtained through a suitable chioce of the input function $v(\cdot)$. The left invertibility can be proven as follows. Let v be a vector of $(C^T U)^{-1}\,\mathcal{V}^*_{\bar{\Sigma}}$ where $\mathcal{V}^*_{\bar{\Sigma}} := \max \mathcal{V}\,(A^T, C^T C\,\mathcal{S}^*_\Sigma, \ker B^T)$. Hence,

$$C^T U\, v \in \mathcal{V}^*_{\bar{\Sigma}}$$

As a consequence, a non-null vector $v' := U\,v$ exists such that $v' \in C^{-T}\,\mathcal{V}^*_{\bar{\Sigma}} = (C\,\mathcal{S}^*_\Sigma)^\perp$. However, v' lies in the range of U, hence it is a vector belonging to $C\,\mathcal{S}^*_\Sigma$. It follows that $v' = v = 0$, and, owing to Property 1, $\bar{\Sigma}$ is left invertible. Now, note that $\mathcal{V}^*_{\bar{\Sigma}} = \mathcal{V}^*_{\Sigma^T}$. Indeed, by the definition of controlled invariant subspace, we only have to prove that

$$\begin{aligned} \mathcal{V}^*_{\Sigma^T} + \operatorname{im} C^T &= \mathcal{V}^*_{\Sigma^T} + C^T C\,\mathcal{S}^*_{\Sigma^T} \\ &= \mathcal{V}^*_{\Sigma^T} + C^T (C^{-T}\,\mathcal{V}^*_{\Sigma^T})^\perp \end{aligned} \tag{7}$$

In fact,

$$\begin{aligned} \operatorname{im} C^T &= C^T\,\mathbb{R}^p = C^T\Big((C^{-T}\,\mathcal{V}^*_{\Sigma^T}) \oplus (C^{-T}\,\mathcal{V}^*_{\Sigma^T})^\perp\Big) \\ &= (\mathcal{V}^*_{\Sigma^T} \cap \operatorname{im} C^T) + C^T (C^{-T}\,\mathcal{V}^*_{\Sigma^T})^\perp \end{aligned}$$

Identity (7) follows by adding $\mathcal{V}^*_{\Sigma^T}$ to both sides of the latter. Hence, it is found that

$$\dim \mathcal{V}^*_{\bar{\Sigma}} = \dim \mathcal{V}^*_{\Sigma^T} = \dim(\mathcal{S}^*_\Sigma)^\perp = n - r$$

However, since $\mathcal{V}^*_{\bar{\Sigma}} \oplus \mathcal{S}^*_{\bar{\Sigma}} = 0_n$, it follows that

$$\dim \mathcal{S}^*_{\bar{\Sigma}} = n - \dim \mathcal{V}^*_{\bar{\Sigma}} = r$$

Now, denote by S and by $\bar{S}$ two basis matrices for $\mathcal{S}^*_\Sigma$ and $\mathcal{S}^*_{\bar{\Sigma}}$, respectively. It is can be shown ([11]) that

$$\mathcal{S}^*_{\widehat{\Sigma}} = \operatorname{im} \begin{bmatrix} S & \times \\ 0 & \bar{S} \end{bmatrix}$$

As a result, the dimension of $\mathcal{S}^*_{\widehat{\Sigma}}$ is $2r$, thus yielding (5). Equation (6) is a straight consequence of the left and right invertibility of the Hamiltonian system. ■

Corollary 2: Let Σ be both left and right invertible. Then

$$\dim \mathcal{V}^*_{\widehat{\Sigma}} = 2 \dim \mathcal{V}^*_\Sigma$$

Proof: The identity follows directly by the left and right invertibility of Σ, expressed by $\mathcal{V}^*_\Sigma \oplus \mathcal{S}^*_\Sigma = 0_n$. ■

Corollary 3: Let Σ be left invertible. The poles of $\mathcal{V}^*_{\widehat{\Sigma}}$ are the invariant zeros of the Hamiltonian system $\widehat{\Sigma}$. These are pairs of the type $(z, -z)$. This set of zeros includes the invariant zeros of Σ.

Proof: The proof follows from Lemma 1, which ensures that $\mathcal{V}^*_{\widehat{\Sigma}}$ has no arbitrarily assignable poles, and from Corollary 1. ■

IV. GEOMETRIC SOLUTION OF THE CHEAP LQ PROBLEM

Theorem 1: Let Assumptions (A1) and (A2) hold. Let Σ be left invertible. An $(n-r)$-dimensional internally stabilizable $(\widehat{A}, \operatorname{im}\widehat{B})$-controlled invariant subspace $\widehat{\mathcal{V}}_R$ contained in $\ker \widehat{C}$ exists, such that all its $n-r$ poles, all unassignable, are stable.

Proof: Since Σ is supposed to be left invertible, Lemma 1 ensures that $\mathcal{V}^*_{\widehat{\Sigma}}$ has no arbitrarily assignable poles. Moreover, from Corollary 1 it follows that the poles of $\mathcal{V}^*_{\widehat{\Sigma}}$ are the invariant zeros of the Hamiltonian system $\widehat{\Sigma}$. These are pairs of the type $(z, -z)$, hence all stable-antistable owing to Assumption (A2). This set of zeros includes the invariant zeros of Σ. Since $\widehat{\Sigma}$ is right invertible, the matrix $T_1 := \begin{bmatrix} T' & T'' \end{bmatrix}$, where $\operatorname{im} T' = \mathcal{V}^*_{\widehat{\Sigma}}$ and $\operatorname{im} T'' = \mathcal{S}^*_{\widehat{\Sigma}}$, is a basis matrix of $\mathbb{R}^{2n}$. Since $\mathcal{V}^*_{\widehat{\Sigma}}$ is an $(\widehat{A}, \operatorname{im}\widehat{B})$-controlled invariant subspace, a matrix $\widehat{F} \in \mathbb{R}^{m \times 2n}$ exists such that $\mathcal{V}^*_{\widehat{\Sigma}}$ is an $(\widehat{A} + \widehat{B}\,\widehat{F})$-invariant subspace. By performing the extended state-space basis transformation defined by T_1, one obtains the following partitioned structure

$$\widehat{A}_{F,T_1} := T_1^{-1}\,(\widehat{A} + \widehat{B}\,\widehat{F})\,T_1 = \begin{bmatrix} M_{SU} & \times \\ 0 & \times \end{bmatrix}$$

where the eigenvalues of $M_{SU} \in \mathbb{R}^{2(n-r) \times 2(n-r)}$ are the poles of $\mathcal{V}^*_{\widehat{\Sigma}}$, all unassignable by virtue of the left invertibility of $\widehat{\Sigma}$, i.e., the strictly stable invariant zeros of the Hamiltonian system and their opposite. Since the subspace of the internal modes of $\mathcal{V}^*_{\widehat{\Sigma}}$ and that of the anti-stable ones are disjoint and both $(\widehat{A} + \widehat{B}\,\widehat{F})$-invariant subspaces, a further basis transformation T_2 in $\mathbb{R}^{2n}$ can be performed so as to split the stable modes from the anti-stable ones. The matrix $\widehat{A}_{F,T_2}$, that corresponds to $\widehat{A}_{F,T_1}$ in this new basis, has the structure

$$\widehat{A}_{F,T_2} := T_2^{-1}\,\widehat{A}_{F,T_1}\,T_2 = \begin{bmatrix} M_S & 0 & \times \\ 0 & M_U & \times \\ 0 & 0 & \times \end{bmatrix}$$

where the eigenvalues of the two submatrices $M_S, M_U \in \mathbb{R}^{(n-r)\times(n-r)}$ are respectively the stable and antistable eigenvalues of the Hamiltonian system $\widehat{\Sigma}$. Hence, the first $n-r$ columns of the matrix $T_1\,T_2$ define a basis for a controlled invariant subspace which is internally stabilizable, and whose poles are exactly the eigenvalues of M_S. This subspace is exactly $\widehat{\mathcal{V}}_R$. ■

Theorem 2: Let Σ be left invertible. Let Assumptions (A1) and (A2) hold. Problem 1 is solvable if and only if

$$x_0 \in \mathcal{P}_{\mathbb{R}^n}(\widehat{\mathcal{V}}_R) \tag{8}$$

i.e., if and only if the initial condition x_0 belongs to the projection of $\widehat{\mathcal{V}}_R$ on the state-space $\mathbb{R}^n$ of Σ. If this condition is met, the state-feedback matrix K solving Problem 1 is given by

$$K = -(F_X + F_\Lambda\,V_\Lambda\,V_X^+)$$

where $V_X, V_\Lambda \in \mathbb{R}^{n \times (n-r)}$ and $F_X, F_\Lambda \in \mathbb{R}^{m \times n}$ are respectively obtained by partitioning a basis matrix $\widehat{V}_R$ of $\widehat{\mathcal{V}}_R$ and a friend $\widehat{F}_R$ of $\widehat{\mathcal{V}}_R$ as

$$\widehat{V}_R = \begin{bmatrix} V_X \\ V_\Lambda \end{bmatrix} \quad \widehat{F}_R = \begin{bmatrix} F_X & F_\Lambda \end{bmatrix}$$

Proof: Necessity and sufficiency of condition (8) can be easily proven by noticing that (8) is equivalent to the existence of a co-state initial condition $\lambda_0 \in \mathbb{R}^n$ such that $\begin{bmatrix} x_0^T & \lambda_0^T \end{bmatrix}^T \in \widehat{\mathcal{V}}_R$. Owing to Theorem 1, an extended state and co-state feedback matrix $\widehat{F}_R$ exists such that the extended state and co-state trajectory obtained by applying the control

$$u(t) = \widehat{F}_R \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \tag{9}$$

entirely lies on $\widehat{\mathcal{V}}_R$ and converges to the origin as t approaches infinity. Hence, the projection of the extended state on $\mathbb{R}^n$ converges to zero as well. Hence, this is a feedback of the extended state which steers the extended state along a stable trajectory evolving in $\widehat{\mathcal{V}}_R$. A function $f : \mathbb{R}_+ \to \mathbb{R}^n$ exists such that for all $t \geq 0$

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} V_X \\ V_\Lambda \end{bmatrix} f(t) \tag{10}$$

Since $\operatorname{im} V_X = \mathcal{P}_{\mathbb{R}^n}(\widehat{\mathcal{V}}_R)$, from (10) it is found that $x(t) \in \operatorname{im} V_X$. Hence, $f(t) = V_X^+ x(t)$ for all $t \geq 0$. The proof ends by writing the optimal feedback control as a function of the sole state $x(t)$:

$$u(t) = (F_X + F_\Lambda V_\Lambda V_X^+)\, x(t)$$

■

V. EXTENSION TO NON LEFT INVERTIBLE SYSTEMS

If system Σ is not left invertible, the results of Theorems 1 and 2 do not directly apply. However, the left invertibility assumption can be easily relaxed, as shown in [9]. Consider to this purpose an auxiliary system $\bar{\Sigma}$, described by the triple $(A + B F, B U, C)$, where

1. F such that $(A + B F)\, \mathcal{V}_\Sigma^* \subseteq \mathcal{V}_\Sigma^*$, and such that all the eigenvalues of $(A + B F)$ restricted to $\mathcal{V}_\Sigma^* \cap \mathcal{S}_\Sigma^*$ (which are freely assignable) are stable
2. U is a basis matrix of $(B^{-1}\, \mathcal{V}_\Sigma^*)^\perp$

The system thus obtained is left invertible, as one can prove by exploiting the dual arguments of those presented in the proof of Lemma 2. Now, let $\bar{K}$ be the optimal state-feedback matrix for the auxiliary system. Then

$$K := U \bar{K} - F$$

is one of the solutions of the original problem. In this case, non uniqueness of the optimal solution is due to the fact that a reachable subspace constrained in $\mathcal{V}_\Sigma^*$ is larger than the sole origin, i.e., a subspace of trajectories that can be followed indefinitely while mantaining $y(t) = 0$, thus not increasing the value of the performance index $J(x, u)$. Hence, different solutions correspond to different choices of matrix F.

VI. CONCLUDING REMARKS

A new approach to the solution of the cheap linear quadratic regulator problem has been presented for continuous-time systems. This method is based on a detailed geometric characterization of the structure of the Hamiltonian system. This insight is achieved by using the standard tools of the geometric approach, without resorting to changes of basis and reduced order algebraic Riccati equations. Thus, the problem is solved by recasting the cheap LQ problem as a perfect decoupling problem in the Hamiltonian system.

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