Geometric Insight Into Discrete-Time Cheap and Singular Linear Quadratic Riccati (LQR) Problems

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***Abstract*—The Hamiltonian system related to discrete-time cheap linear quadratic Riccati (LQR) problems is analyzed in a purely geometric context, with the twofold purpose of getting a useful insight into its structural features and deriving a numerically implementable solution for the infinite-horizon case by only using the standard geometric approach routines available.**

***Index Terms*—Cheap control, computational methods, geometric approach, linear-quadratic (LQ) problems.**

I. Introduction

Cheap and singular linear quadratic (LQ) optimal control problems have been widely investigated since the beginning of optimal control theory and a large number of contributions can be found in the literature. As far as computational procedures are concerned, at present there are two main approaches, one using matrix pencils and the other based on linear matrix inequalities. Regarding the matrix pencil approach, relevant contributions were made by Arnold and Laub [1], Van Doren [2], and Ionescu et al. [3]–[5], who recently also gave an exhaustive survey of this stream of research in [6] and [7]. Geerts in [8], Saberi, Sannuti, Chen and Stoorvogel in [9]–[13] analyzed cheap and singular LQ optimal control by means of linear matrix inequalities (LMIs). The LMI and the matrix pencil approaches have been developed for both discrete and continuous time systems. The algebraic Riccati equation is stated in a generalized form and therefore can handle regular, singular and cheap LQ control problems. The generalized algebraic Riccati equation is solved by a rank-minimizing solution of an associated linear matrix inequality or by an invariant subspace of a simplectic pencil associated to the generalized algebraic Riccati equation (ARE). Connections between the two approaches were pointed out in [14].

The LQ problem was also studied in a geometric framework by Silverman, Hautus, Willems and Kìtapçi in [15]–[17]. In these papers the system structure is analyzed through geometric tools based on the notion of weak unobservability and almost invariant subspaces. A geometric algorithm allows singular problems to be stated in standard nonsingular forms, leading to reduced order AREs. In [17] the authors analyze algorithms for nonminimum phase continuous-time systems. The connection between cheap control and perfect tracking was considered by Francis in [18].

In this paper the LQ optimal control problem is studied for general discrete-time systems, no matter if they are nonleft-invertible or nonminimum-phase. A solution to the cheap and singular LQ problem that is alternative to those investigated in the literature is proposed. Our approach does not require the solution of any ARE, or LMI, or the computation of deflating subspaces of suitable matrix pencils, but it only uses the basic tools of the geometric approach, [19]–[21] by which the Hamiltonian system is analyzed and solved.

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The analysis of the Hamiltonian system properties that are relevant for the cheap control problem is carried out with general system theory concepts, such as duality and invertibility, for systems described by triples (A, B, C), without any feedthrough matrix D. This simplified approach avoids the use of more sophisticated and complex tools, like almost controlled invariants and output nulling subspaces (also called strong unobservability subspaces in the most recent literature) and their duals. Restricting to the cheap control problem which is the simplest formulation has the advantage of being closely related to optimal and perfect tracking problems, while, on the other hand, it does not affect generality. In fact, the result derived for the cheap problem can also be applied to the singular and regular cases by a straightforward space extension. Hence, in this geometric setting, cheap control is considered to be the most general one. Reduction of quadruples to triples by state extension is a convenient way, [19], of avoiding involved algorithms while preserving generality.

A further significant advantage of treating the overall Hamiltonian system with standard geometric tools is that non left-invertible systems are easily handled.

II. Statement of the Problem and Background

In order to concisely state the stability properties, the following notation will be assumed for the most frequently used subsets of the complex field $\mathbb{C}$: $\mathbb{C}^{\odot}, \mathbb{C}^{\circledcirc}, \mathbb{C}^{\otimes}$ and $\mathbb{C}^{\circ}$ stand for the open unit disc in the complex plane, the open unit disc without the origin, the open set of complex numbers outside the unit disc and the unit circle, respectively. Thus the dynamic matrix A is said to be $\mathbb{C}^{\circledcirc}$-stable ($\mathbb{C}^{\odot}$-stable) if all its eigenvalues are in $\mathbb{C}^{\circledcirc}$ (in $\mathbb{C}^{\odot}$) and the pair (A, B) is said to be $\mathbb{C}^{\circledcirc}$-stabilizable ($\mathbb{C}^{\odot}$-stabilizable) if all the uncontrollable modes of (A, B) are in $\mathbb{C}^{\circledcirc}$ (in $\mathbb{C}^{\odot}$).

Consider the linear discrete-time-invariant system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \quad x(0) = x_0, k = 0, 1, \ldots \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$ denote the state, the control input and the controlled output, respectively. The matrices B and C are assumed to be full rank. System (1) will also be referred to as the triple (A, B, C).

The following notation is required for the geometric approach: $\mathcal{B}$ stands for the image of $B(\operatorname{im} B), \mathcal{C}$ for the null space of $C(\ker C), \mathcal{V}^*$ for the maximum $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}(\max \mathcal{V}(A, \mathcal{B}, \mathcal{C})), \mathcal{S}^*$ for the minimum $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{B}(\min \mathcal{S}(A, \mathcal{C}, \mathcal{B}))$ and $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$ for the reachable set on $\mathcal{V}^*$.

Finally, let us recall that the set of the invariant zeros of system (1), i.e., the set of the internal unassignable eigenvalues of $\mathcal{V}^*$, is defined as $\sigma(A+BF)_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}$, where $\sigma(A+BF)$ is the spectrum of matrix $A+BF$ with F such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$.

The cheap discrete-time LQR problem is stated as follows.

Problem 1: Refer to system (1) and assume that

1) the pair (A, B) is $\mathbb{C}^{\odot}$-stabilizable;
2) the triple (A, B, C) has no invariant zeros in $\mathbb{C}^{\circ}$.

Determine the state feedback matrix $K \in \mathbb{R}^{p \times n}$ such that

1) $A - BK$ is $\mathbb{C}^{\odot}$-stable;
2) the corresponding state trajectory minimizes the performance index

$$J := \sum_{k=0}^{\infty} y(k)^T y(k). \tag{2}$$

The following standard properties are recalled for the reader's convenience.

Property 1: System (1) is left (right) invertible if and only if $\mathcal{V}^* \cap \mathcal{S}^* = \{0\} (\mathcal{S}^* + \mathcal{V}^* = \mathbb{R}^n)$.

Property 2: If system (1) is right-invertible but not left-invertible, the input sequence corresponding to a given output sequence can only be determined modulo $B^{-1}\mathcal{V}^*$, the inverse image[1] of $\mathcal{V}^*$ with respect to the linear transformation B. If system (1) is left-invertible but not right-invertible, the output sequence can be imposed modulo any complement of $C\mathcal{S}^*$.

Proofs of Properties 2 can be found in [21] and [22] for continuous-time systems. Their extension to the discrete-time case is straightforward. In the following, left and right invertibility of adjoint and reverse-time systems are discussed.

The reverse-time representation of system (1) is defined as follows. If A is nonsingular, the state dynamics can also be written as a backward recursion with the output equation accordingly modified. Thus the system equations can be written as

$$\begin{aligned} x(k) &= A^{-1}x(k+1) - A^{-1}B\,u(k) \\ y(k) &= CA^{-1}x(k+1) - CA^{-1}B\,u(k). \end{aligned}$$

The quadruple $(A^{-1}, -A^{-1}B, CA^{-1}, -CA^{-1}B)$, namely, the system

$$\begin{aligned} z(k+1) &= A^{-1}z(k) - A^{-1}Bv(k) \\ w(k) &= CA^{-1}z(k) - CA^{-1}Bv(k) \end{aligned} \tag{3}$$

will be referred to as the reverse-time system associated to system (1).

Property 3: System (1) is left-invertible (right-invertible) if and only if its adjoint is right-invertible (left-invertible). Moreover, system (1) is right-invertible (left-invertible) if and only if the associated reverse-time system is right-invertible (left-invertible).

III. Geometric Features of the Cheap LQR Problem

The geometric insight to the cheap control Problem 1 is obtained by analyzing the geometric properties of the Hamiltonian system and algebraic condition, see, e.g., [23]

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ p(k) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 2C^TC & A^T \end{bmatrix} \begin{bmatrix} x(k) \\ p(k+1) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) \\ 0 &= [0 \quad B^T] \begin{bmatrix} x(k) \\ p(k+1) \end{bmatrix} \end{aligned} \tag{4}$$

where p denotes the costate.

Assumption 1: Assume that the pair (A, B) is $\mathbb{C}^{\circledcirc}$-stabilizable, i.e., it does not have uncontrollable modes in zero.

Remark 1: Non singularity of matrix A can be assumed with no loss of generality owing to Assumption 1. In fact, in the case of A singular, a state feedback $u(k) = Hx(k)$ can be performed in order to obtain a nonsingular system matrix $\bar{A} := A + BH$. Then, the original problem solution is $K = \bar{K} - H$, where $\bar{K}$ denotes the state feedback matrix solving the problem referred to the triple $(\bar{A}, B, C)$.

Under Assumption 1, the forward recursion $p(k+1) = -2A^{-T}C^TCx(k) + A^{-T}p(k)$ enables writing (4) as

$$\begin{aligned} z(k+1) &= \hat{A}z(k) + \hat{B}u(k) \\ 0 &= \hat{C}z(k) \end{aligned} \tag{5}$$

[1]For computational purposes, recall the identity $B^{-1}\mathcal{V}^* = (B^T\mathcal{V}^{*\perp})^\perp$.

with $z(k) := [x(k)^T \; p(k)^T]^T$ and

$$\begin{aligned} \hat{A} &:= \begin{bmatrix} A & 0 \\ -2A^{-T}C^TC & A^{-T} \end{bmatrix} \quad \hat{B} := \begin{bmatrix} B \\ 0 \end{bmatrix} \\ \hat{C} &:= [-2B^TA^{-T}C^TC \quad B^TA^{-T}]. \end{aligned}$$

Conversely, by using the backward recursion $x(k) = A^{-1}x(k+1) - A^{-1}Bu(k)$, (4) can also be written as

$$\begin{aligned} z(k) &= \hat{A}_1\,z(k+1) + \hat{B}_1\,u(k) \\ 0 &= \hat{C}_1 z(k+1) \end{aligned} \tag{6}$$

where

$$\begin{aligned} \hat{A}_1 &:= \begin{bmatrix} A^{-1} & 0 \\ 2C^TCA^{-1} & A^T \end{bmatrix} = \hat{A}^{-1} \\ \hat{B}_1 &:= \begin{bmatrix} -A^{-1}B \\ -2C^TCA^{-1}B \end{bmatrix} = -\hat{A}^{-1}\hat{B} \\ \hat{C}_1 &:= [0 \quad B^T] = \hat{C}\hat{A}^{-1}. \end{aligned}$$

From now on, the triples $(\hat{A}, \hat{B}, \hat{C})$ and $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ will be referred to as "the Hamiltonian system" and "the reverse-time Hamiltonian system", respectively. The analysis of their structural properties, carried out by the standard tools of the geometric approach, will lead to the solution of Problem 1 and provide the insight this papers aims at. Symbols $\hat{\mathcal{B}} := \operatorname{im}\hat{B}, \hat{\mathcal{B}}_1 := \operatorname{im}\hat{B}_1,\ \hat{\mathcal{C}} := \ker\hat{C}, \hat{\mathcal{C}}_1 := \ker\hat{C}_1, \hat{\mathcal{V}}^* := \max, \mathcal{V}(\hat{A}, \hat{\mathcal{B}}, \hat{\mathcal{C}}), \hat{\mathcal{V}}_1^* := \max, \mathcal{V}(\hat{A}_1, \hat{\mathcal{B}}_1, \hat{\mathcal{C}}_1), \hat{\mathcal{S}}^* := \min\,\mathcal{S}(\hat{A}, \hat{\mathcal{C}}, \hat{\mathcal{B}}), \hat{\mathcal{S}}_1^* := \min,$ and $\mathcal{S}(\hat{A}_1, \hat{\mathcal{C}}_1, \hat{\mathcal{B}}_1)$ will be used, consistently with the notation previously introduced.

The following lemmas are functional to the proof of the main theorem, presented in Section IV. All the results are derived under left-invertibility assumption of the triple (A, B, C), which will be removed in Remark 2.

Lemma 1: Let the triple (A, B, C) be left-invertible. Then $(\hat{A}, \hat{B}, \hat{C})$ and $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ are both right and left-invertible, i.e.,

$$\hat{\mathcal{V}}^* \oplus \hat{\mathcal{S}}^* = \mathbb{R}^{2n}, \tag{7}$$

$$\hat{\mathcal{V}}_1^* \oplus \hat{\mathcal{S}}_1^* = \mathbb{R}^{2n}. \tag{8}$$

Proof: First, let us note that the triple $(\hat{A}, \hat{B}, \hat{C})$ is the series connection of system (1), here briefly denoted by Σ_1, and

$$\begin{aligned} p(k+1) &= A^{-T}p(k) - 2A^{-T}C^Ty(k) \\ y(k) &= B^TA^{-T}p(k) - 2B^TA^{-T}C^Ty(k) \end{aligned}$$

referred to as Σ_2. Let us also note that Σ_2 is the reverse-time system associated to the adjoint of Σ_1 except for an input scaling by a factor 2. Hence, Σ_2 is right-invertible owing to Property 2. Since Σ_1 is left-invertible but, in general, not right-invertible, owing to Property 2 it is possible to impose any projection of the output of Σ_1 on the output reachability subspace, while the complementary projection cannot be imposed.

On the other hand, since Σ_2 is right-invertible but, in general, not left-invertible, owing to Property 2 it is possible to recognize the projection of the input of Σ_2 on any complement of the input unobservability subspace (IUS), while the projection of the input on the IUS itself cannot be observed.

Since the IUS of Σ_2 is a complement of the ORS of Σ_1, a one-to-one mapping exists between the input sequences of Σ_1 and the output sequences of Σ_2, which proves the right invertibility of the triple $(\hat{A}, \hat{B}, \hat{C})$. Owing to Property 2, the triple $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ is also right-invertible, since it is the reverse-time representation of

$(\hat{A}, \hat{B}, \hat{C})$. Finally, let us note that $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ is a state-space representation of the adjoint of system $(\hat{A}, \hat{B}, \hat{C})$. In fact, the triple $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ can be obtained from $(\hat{A}^T, \hat{C}^T, \hat{B}^T)$ by the state-space basis transformation

$$T = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Then, $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ and $(\hat{A}, \hat{B}, \hat{C})$ are left-invertible owing to Property 3. ∎

Lemma 2: Let the triple (A, B, C) be left-invertible and r be defined as

$$r := \dim \hat{\mathcal{S}}^*. \tag{9}$$

Hence, the following equalities hold:

$$\dim \hat{\mathcal{V}}^* = 2n - r \tag{10}$$
$$\dim \hat{\mathcal{V}}_1^* = 2\,n - r \tag{11}$$
$$\dim \hat{\mathcal{S}}_1^* = r. \tag{12}$$

Proof: Owing to Lemma 1, $(\hat{A}, \hat{B}, \hat{C})$ and $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ are both right and left-invertible and (7) and (8) hold. Equations (7) and (9) imply (10). Owing to the duality properties of controlled and conditioned invariants, it follows that:

$$\begin{aligned} \min \mathcal{S}(A, \mathcal{C}, \mathcal{B}) &= (\max\ \mathcal{V}(A^T, \mathcal{C}^\perp, \mathcal{B}^\perp))^\perp \\ &= (\max\ \mathcal{V}(A^T, \operatorname{im} C^T, \ker B^T))^\perp. \end{aligned} \tag{13}$$

Since $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ is a state-space representation of the adjoint of $(\hat{A}, \hat{B}, \hat{C})$ and the dimensions of the maximum controlled invariants and minimum conditioned invariants are preserved under change of basis in the state space, (8) and (13) prove (11) and (12). ∎

In the following lemmas, the internal eigenvalues of $\hat{\mathcal{V}}^*$ and those of $\hat{\mathcal{V}}_1^*$ will be considered. It is worth noting that these are all unassignable owing to left invertibility of the triples $(\hat{A}, \hat{B}, \hat{C})$ and $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$, respectively. So the expressions "internal unassignable eigenvalues" and "internal eigenvalues" will be used indifferently.

Lemma 3: The subspace $\hat{\mathcal{S}}_1^*$ is an $(\hat{A}, \hat{B})$-controlled invariant contained in $\hat{\mathcal{C}}$, hence

$$\hat{\mathcal{S}}_1^* \subseteq \hat{\mathcal{V}}^*. \tag{14}$$

The r internal unassignable eigenvalues of $\hat{\mathcal{S}}_1^*$ are all equal to zero, hence the triple $(\hat{A}, \hat{B}, \hat{C})$ has r invariant zeros equal to zero. A similar result holds for the subspace $\hat{\mathcal{S}}^*$ and the triple $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$.

Proof: Let us recall the standard algorithm for conditioned invariants, see [24] and [19, p. 208]. The subspace $\hat{\mathcal{S}}_1^*$ is the last term of the sequence

$$\begin{aligned} &\hat{\mathcal{S}}_{1,0} = \hat{\mathcal{B}}_1 \\ &\hat{\mathcal{S}}_{1,i} = \hat{A}_1(\hat{\mathcal{S}}_{1,i-1} \cap \hat{\mathcal{C}}_1) + \hat{\mathcal{B}}_1, \quad i = 1, \ldots, k \end{aligned} \tag{15}$$

where the value of k is given by the condition $\hat{\mathcal{S}}_{1,k+1} = \hat{\mathcal{S}}_{1,k}$. In other words, $\hat{\mathcal{S}}_1^*$ is the locus of the states of $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ that can be reached from the origin in a finite number of steps along trajectories belonging to $\hat{\mathcal{C}}_1$ until the last but one step. Then, any state of $\hat{\mathcal{S}}_1^*$ can be driven to the origin (in the forward system) along a trajectory completely belonging to $\hat{\mathcal{S}}_1^*$ (the same trajectory followed backward). Thus, $\hat{\mathcal{S}}_1^*$ is an $(\hat{A}, \hat{B})$-controlled invariant. Let us now show that $\hat{\mathcal{S}}_1^*$ is contained in $\hat{\mathcal{C}}$. Since $\hat{\mathcal{S}}_1^* = \hat{A}_1(\hat{\mathcal{S}}_1^* \cap \hat{\mathcal{C}}_1) + \hat{\mathcal{B}}_1$, the inclusion $\hat{\mathcal{S}}_1^* \subseteq \hat{\mathcal{C}}$ is equivalent to

$$\hat{C}(\hat{A}_1(\hat{\mathcal{S}}_1^* \cap \hat{\mathcal{C}}_1) + \hat{\mathcal{B}}_1) = 0$$

which can be proven by considering that the linear transformation of subspaces is distributive with respect to the sum and that

$$\hat{C}\hat{A}_1(\hat{\mathcal{S}}_1^* \cap \hat{\mathcal{C}}_1) \subseteq \hat{C}\hat{A}_1\hat{\mathcal{C}}_1 = \hat{C}\hat{A}_1 \ker(\hat{C}\hat{A}_1) = 0$$

and

$$\hat{C}\hat{\mathcal{B}}_1 = \hat{C} \operatorname{im} \mathcal{B}_1 = \operatorname{im}(\hat{C}B_1) = 0.$$

Therefore, $\hat{\mathcal{S}}_1^*$ is contained in $\hat{\mathcal{V}}^*$, the maximum $(A, \mathcal{B})$-controlled invariant contained in $\hat{\mathcal{C}}$. Since any state in $\hat{\mathcal{S}}_1^*$ is driven to the origin in a finite number of steps, it follows that the r internal eigenvalues of $\hat{\mathcal{S}}_1^*$ are equal to zero. ∎

Lemma 4: The subspace

$$\hat{\mathcal{V}}_S := \hat{\mathcal{V}}^* \cap \hat{\mathcal{V}}_1^* \tag{16}$$

is both an $(\hat{A}, \hat{B})$ and an $(\hat{A}_1, \hat{B}_1)$-controlled invariant. Its dimension is $2(n-r)$. Its internal eigenvalues are the nonnull invariant zeros of $(\hat{A}, \hat{B}, \hat{C})$. These are in pairs of the type $(z_i, 1/z_i)$, i.e., stable/unstable. This set of zeros includes the invariant zeros of the triple (A, B, C).

Proof: Since $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ is the adjoint of $(\hat{A}, \hat{B}, \hat{C})$, the set of the invariant zeros of $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$, i.e., the set of the internal eigenvalues of $\hat{\mathcal{V}}_1^*$, coincides with the set of the invariant zeros of $(\hat{A}, \hat{B}, \hat{C})$, i.e., the set of the internal eigenvalues of $\hat{\mathcal{V}}^*$. Since $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$ is the reverse-time system associated to $(\hat{A}, \hat{B}, \hat{C})$, the set of the nonnull invariant zeros of $(\hat{A}_1, \hat{B}_1, \hat{C}_1)$—in number of $2(n-r)$ as a consequence of Lemma 3—coincides with the set of the nonnull invariant zeros of $(\hat{A}, \hat{B}, \hat{C})$. Hence, this set counts $2(n-r)$ elements, each paired with its inverse. The internal structure of the triple $(\hat{A}, \hat{B}, \hat{C})$ ensures that this set also includes all the nonnull internal eigenvalues of $\mathcal{V}^* = \max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$ and their inverses. Finally, since any nonnull internal eigenvalue of $\hat{\mathcal{V}}^*$ is also an internal eigenvalue of $\hat{\mathcal{V}}_1^*$, the subspace $\hat{\mathcal{V}}_S$ defined in (16) has $2(n-r)$ internal eigenvalues, hence dimension $2(n-r)$. Furthermore, $\hat{\mathcal{V}}_S$ is both an $(\hat{A}, \hat{B})$ and an $(\hat{A}_1, \hat{B}_1)$-controlled invariant since any trajectory associated to a nonnull internal eigenvalue of $\hat{\mathcal{V}}^*$ is also a trajectory belonging to $\hat{\mathcal{V}}_1^*$. ∎

IV. Geometric Solution of the Cheap LQR Problem

A geometric solution to Problem 1 is provided. The following statements, whose proofs are constructive, are based on lemmas presented in Section III.

Theorem 1: Refer to the Hamiltonian system (5), namely the triple $(\hat{A}, \hat{B}, \hat{C})$, and assume that (A, B, C) is left-invertible and with no invariant zeros in $\mathbb{C}^\circ$. There exists an internally stabilizable $(\hat{A}, \hat{B})$-controlled invariant $\hat{\mathcal{V}}_R$ contained in $\hat{\mathcal{C}}$ whose dimension is n. $\hat{\mathcal{V}}_R$ has n unassignable eigenvalues, r of which are null and $n-r$ are not. The latter are the nonnull invariant zeros of $(\hat{A}, \hat{B}, \hat{C})$ in $\mathbb{C}^\odot$.

Proof: From Lemma 1 [(8)], Lemma 3 [(14)] and Lemma 4 [(16)], it follows that:

$$\hat{\mathcal{V}}^* = \hat{\mathcal{S}}_1^* \oplus \hat{\mathcal{V}}_S. \tag{17}$$

Hence, owing to (7) in Lemma 1, the matrix $T_0 := [T_1 \quad T_2 \quad T_3]$, where T_1, T_2 and T_3 are such that $\hat{\mathcal{S}}_1^* = \operatorname{im} T_1, \hat{\mathcal{V}}_S = \operatorname{im} T_2$ and $\hat{\mathcal{S}}^* = \operatorname{im} T_3$, respectively, is a basis matrix of $\mathbb{R}^{2n}$. Let us also note that, since $\hat{\mathcal{V}}^*$ is an $(\hat{A}, \hat{B})$-controlled invariant, at least one matrix $\hat{F}$ exists such that $\hat{\mathcal{V}}^*$ is an $(\hat{A} + \hat{B}\hat{F})$-invariant. Let us define $\hat{A}_F := \hat{A} + \hat{B}\hat{F}$, and perform the state-space basis transformation defined by T_0. The matrix $\hat{A}_{F,T_0}$ which corresponds to $\hat{A}_F$ in the new basis, partitioned according to T_0, has the structure

$$\hat{A}_{F,T_0} := T_0^{-1}\hat{A}_F\, T_0 = \left[\begin{array}{cc|c} A_N & 0 & \times \\ 0 & A_{S,U} & \times \\ \hline 0 & 0 & \times \end{array}\right]$$

where $A_N \in \mathbb{R}^{r\times r}$ and $A_{S,U} \in \mathbb{R}^{2(n-r)\times 2(n-r)}$. The $2n-r$ eigenvalues of the submatrix in the top left corner of $\hat{A}_{F,T_0}$ are the unassignable internal eigenvalues of $\hat{\mathcal{V}}^*$, i.e., the invariant zeros of $(\hat{A},\hat{B},\hat{C})$. Owing to Lemma 3, the r eigenvalues of A_N are all equal to zero, i.e., A_N is nilpotent and its eigenvalues are the null invariant zeros of $(\hat{A},\hat{B},\hat{C})$. Owing to Lemma 4, the $2(n-r)$ eigenvalues of $A_{S,U}$ are the $n-r$ invariant zeros of $(\hat{A},\hat{B},\hat{C})$ in $\mathbb{C}^{\circledcirc}$ and their inverses. Since the subspace of the stable modes of $\hat{\mathcal{V}}^*$ and that of the unstable ones are disjoint and both $(\hat{A}+\hat{B}\hat{F})$-invariant, it is possible to perform a further state-space basis transformation T_1 whose aim is to separate the stable modes from the unstable ones. The matrix $\hat{A}_{F,T_1}$ that corresponds to $\hat{A}_{F,T_0}$ in the new basis has the structure

$$\hat{A}_{F,T_1} := T_1^{-1}\hat{A}_{F,T_0}\, T_1 = \begin{bmatrix} A_N & 0 & 0 & \times \\ 0 & A_S & 0 & \times \\ 0 & 0 & A_U & \times \\ 0 & 0 & 0 & \times \end{bmatrix}$$

where $A_S, A_U \in \mathbb{R}^{(n-r)\times(n-r)}$. The $n-r$ eigenvalues of A_S are the invariant zeros of $(\hat{A},\hat{B},\hat{C})$ in $\mathbb{C}^{\circledcirc}$, while the $n-r$ eigenvalues of A_U are the invariant zeros of $(\hat{A},\hat{B},\hat{C})$ in $\mathbb{C}^{\otimes}$. To complete the proof, let us denote by T the product of the matrices T_1 and T_0 of the changes of basis, i.e., $T := T_1T_0$. The first n column vectors of T define a basis matrix $\hat{V}_R$ for the subspace $\hat{\mathcal{V}}_R$ that meets the conditions of the statement. ■

Corollary 1: Let us refer to Problem 1 and assume that the triple (A,B,C) is left-invertible and that the pair (A,B) is $\mathbb{C}^{\circledcirc}$-stabilizable, i.e., with no uncontrollable modes in zero (Assumption 1). The state feedback matrix solving the problem is unique and given by

$$K = -(F_X + V_P V_X^{-1} F_P) \tag{18}$$

where $V_X, V_P \in \mathbb{R}^{n\times n}$ are obtained by partitioning $\hat{V}_R$ as

$$\hat{V}_R = \begin{bmatrix} V_X \\ V_P \end{bmatrix}$$

and F_X, $F_P \in \mathbb{R}^{p\times n}$ by partitioning $\hat{F}$ accordingly.

Proof: The existence of a solution to Problem 1 implies that, for any initial condition x_0, a costate initial condition $p_0 = V_P V_X^{-1} x_0$ exists such that the extended state $z_0 = [x_0^T \quad p_0^T]^T$ at $k=0$ belongs to $\hat{\mathcal{V}}_R$. According to Theorem 1, the feedback control solving Problem 1 is given by

$$u(k) = \hat{F} z(k). \tag{19}$$

This is a feedback of the extended state $z(k)$ which steers the extended state along a stable trajectory evolving on $\hat{\mathcal{V}}_R$. Left invertibility of the Hamiltonian system $(\hat{A},\hat{B},\hat{C})$ implies uniqueness of the solution to Problem 1. The proof ends by writing the feedback control (19) as a function of the sole state x:

$$u(k) = \hat{F}\begin{bmatrix} x(k) \\ p(k) \end{bmatrix} = \left(F_X + F_P V_P V_X^{-1}\right) x(k).$$

■

A. Uncontrollable Systems

Assumption 1 in Corollary 1 is technical in nature and can easily be relaxed at the expense of a more involved discussion. This paragraph shows how to relax Assumption 1. The proofs of the statements are similar to those of Theorem 1 and Corollary 1 and are omitted.

Consider the uncontrollable system with $x_c \in \mathbb{R}^c$ and $x_u \in \mathbb{R}^{n-c}$

$$\begin{bmatrix} x_c(k+1) \\ x_u(k+1) \end{bmatrix} = \begin{bmatrix} A_c & A_{cu} \\ 0 & A_u \end{bmatrix}\begin{bmatrix} x_c(k) \\ x_u(k) \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u(k)$$
$$y = [C_c \quad C_u]\begin{bmatrix} x_c(k) \\ x_u(k) \end{bmatrix}. \tag{20}$$

The Hamiltonian system and the algebraic condition can be derived disregarding the costate corresponding to the uncontrollable part of the system

$$\begin{aligned} z_r(k+1) &= \hat{A}_r z_r(k) + \hat{B}_r\, u(k) \\ 0 &= \hat{C}_r z_r(k) \end{aligned} \tag{21}$$

with $z_r(k) := [x_c(k)^T \quad x_u(k)^T \quad p_c(k)^T]^T$ and

$$\begin{aligned} \hat{A}_r &:= \begin{bmatrix} A_c & A_{cu} & 0 \\ 0 & A_u & 0 \\ -2A_c^{-T}C_c^TC_c & -2A_c^{-T}C_c^TC_u & A_c^{-T} \end{bmatrix} \\ \hat{B}_r^T &:= [B_c^T \quad 0 \quad 0] \\ \hat{C}_r &:= B_c^T\,[-2A_c^{-T}C_c^TC_c \quad -2A_c^{-T}C_c^TC_u \quad A_c^{-T}]. \end{aligned}$$

The dynamics of the costate $p_u(k) \in \mathbb{R}^{n-c}$ is not taken into account in (21) since it does not excite the dynamics of $x(k)$ and $p_c(k)$.

Note that in this reduced formulation, only invertibility of the controllable part of the dynamic matrix is required. This is ensured by Remark 1. Finally, Theorem 1 and Corollary 1 are extended as follows.

Theorem 2: Refer to $(\hat{A}_r,\hat{B}_r,\hat{C}_r)$, and assume that (A,B) is $\mathbb{C}^{\odot}$-stabilizable, left-invertible and with no zeros in $\mathbb{C}^{\circ}$. There exists an internally stabilizable $(\hat{A}_r,\hat{B}_r)$-controlled invariant $\hat{\mathcal{V}}_r$ contained in $\ker\hat{C}_r$ whose dimension is n.

Corollary 2: Refer to Problem 1 and assume that the triple (A,B,C) is left-invertible. The state feedback matrix that solves the problem is unique and is given by

$$K = -\left(F_X + F_P V_P\, V_X^{-1}\right) \tag{22}$$

where $V_X, V_P \in \mathbb{R}^{n\times n}$ are obtained by partitioning $\hat{V}_r$, the basis matrix of $\hat{\mathcal{V}}_r$, as

$$\hat{V}_R = \begin{bmatrix} V_X \\ V_P \end{bmatrix}$$

and $F_X, F_P \in \mathbb{R}^{p\times n}$ are obtained by partitioning $\hat{F}$ accordingly, with $\hat{F}$ such that $(\hat{A}_r + \hat{B}_r\hat{F})\hat{\mathcal{V}}_r \subseteq \hat{\mathcal{V}}_r$.

V. Non Left-Invertible Systems

If the triple (A,B,C) is nonleft-invertible, the results of Theorems 1 and 2 and Corollaries 1 and 2 do not apply. The left-invertibility assumption can easily be relaxed at the expense of some additional algebra, as shown in the following.

Remark 2: Let the triple (A,B,C) be non left-invertible. Consider the auxiliary left-invertible system $(A+BF, BU, C)$, such that $(A+BF)\mathcal{V}^*\subseteq\mathcal{V}^*$ with $\sigma(A+BF)\,|_{\mathcal{R}_{\mathcal{V}^*}}$ stable, and denote by U a basis matrix of $\mathcal{U} := (B^{-1}\mathcal{V}^*)^{\perp}$, the orthogonal complement of the inverse

TABLE I
COORDINATE TRANSFORMATION FOR NON LEFT INVERTIBLE SYSTEMS.

	T_1	T_2	T_3	T_4	T_5
$\mathcal{R}_{\hat{\mathcal{V}}^*}$			•		
$\mathcal{R}_{\hat{\mathcal{V}}_1^*}$			•		
$\hat{\mathcal{S}}_1^*$	•		•		
$\hat{\mathcal{S}}^*$			•		•
$\hat{\mathcal{V}}^*$	•	•	•	•	
$\hat{\mathcal{V}}_1^*$		•	•	•	•

image of $\mathcal{V}^*$ with respect to B. Let K_L be the optimal state feedback matrix for the auxiliary system. Then

$$K := UK_L - F \tag{23}$$

is one of the solutions of the original problem. Let us recall that the solution to Problem 1 is not unique for non left-invertible systems. Different solutions correspond to different choices of $\sigma(A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}$.

This procedure is straightforward but requires some preliminary algebra. A different approach to solving the cheap control problem for nonleft-invertible systems consists in directly manipulating the Hamiltonian system stated for the nonleft-invertible triple.

The algebraic manipulation of the basis matrices of the Hamiltonian system is more involved in this case. It can be shown that

$$\mathcal{R}_{\hat{\mathcal{V}}^*} = \mathcal{R}_{\hat{\mathcal{V}}_1^*} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \in \mathcal{R}_{\mathcal{V}^*}, y = 0 \right\}.$$

Moreover, let $c := \dim \mathcal{R}_{\hat{\mathcal{V}}^*}$ and r such that $r + c = \dim \hat{\mathcal{S}}_1^*$, it can be proven that the nonnull invariant zeros of $(\hat{A}, \hat{B}, \hat{C})$ are $2(n-r-c)$, each in pair with its inverse. Then, similarly to Theorem 1, the following remark applies.

Remark 3: Refer to the Hamiltonian system (5), and assume that the triple (A, B, C) is not left-invertible and with no invariant zeros in $\mathbb{C}^\circ$.

Perform the coordinate transformation, described in Table I

$$T := [T_1, T_2, T_3, T_4, T_5, T_6]$$

where

$$\begin{aligned} \operatorname{im} T_3 &= \mathcal{R}_{\hat{\mathcal{V}}^*} \\ \operatorname{im}[T_1, T_3] &= \hat{\mathcal{S}}_1^* \\ \operatorname{im}[T_1, T_5] &= \hat{\mathcal{S}}^* \\ \operatorname{im}[T_1, T_2, T_3, T_4] &= \hat{\mathcal{V}}^* \\ \operatorname{im}[T_2, T_3, T_4, T5] &= \hat{\mathcal{V}}_1^* \end{aligned}$$

and $T_2(T_4)$, whose dimension is $n-r-c$, account for stable (unstable) zeros of the Hamiltonian system. Then, the $(\hat{A}, \hat{B})$-controlled invariant subspace

$$\hat{\mathcal{V}}_R := \operatorname{im}[T_1, T_2, T_3] \tag{24}$$

is internally stabilizable, contained in $\hat{\mathcal{C}}$ and has dimension n. The internal unassignable eigenvalues of $\hat{\mathcal{V}}_R$ consist of r null eigenvalues, c assignable eigenvalues and $n-r-c$ nonnull eigenvalues, which are the nonnull invariant zeros of $(\hat{A}, \hat{B}, \hat{C})$ in $\mathbb{C}^\odot$.

Now, a state feedback matrix $\hat{F}$, making the $(\hat{A}, \hat{B})$-controlled invariant $\hat{\mathcal{V}}^*$ an $(\hat{A}+\hat{B}\hat{F})$ invariant, can be computed such that the internal assignable eigenvalues are stable. Matrix $\hat{A}+\hat{B}\hat{F}$, partitioned according to T, in the new basis assumes the structure

$$T^{-1}(\hat{A}+\hat{B}\hat{F})T = \begin{bmatrix} A_N & 0 & 0 & 0 & \times \\ 0 & A_S & 0 & 0 & \times \\ 0 & 0 & A_C & 0 & \times \\ 0 & 0 & 0 & A_U & \times \\ 0 & 0 & 0 & 0 & \times \end{bmatrix} \tag{25}$$

where $A_N \in \mathbb{R}^{r\times r}, A_R \in \mathbb{R}^{c\times c}$ and $A_S,\ A_U \in \mathbb{R}^{(n-r-c)\times(n-r-c)}$. A_N is nilpotent and its eigenvalues are the null invariant zeros of the Hamiltonian system. A_C accounts for the c internal assignable eigenvalues of the system and finally, the $n-r-c$ eigenvalues of A_S are the invariant zeros of $(\hat{A}, \hat{B}, \hat{C})$ in $\mathbb{C}^{\circledcirc}$, while the $n-r-c$ eigenvalues of A_U are the invariant zeros of $(\hat{A}, \hat{B}, \hat{C})$ in $\mathbb{C}^{\otimes}$.

Finally, under Assumption 1, the solution to Problem 1 for non left-invertible systems is given by Corollary 1 provided that the subspace $\hat{\mathcal{V}}_R$ and the state feedback matrix $\hat{F}$ are evaluated according to (24) and (25).

As for Remark 2, here solutions to the cheap control problem are parameterized by the set of eigenvalues assigned on the reachable set on the maximum $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}(\operatorname{im} T_3)$.

It is worth noting that, although this approach to solving cheap control problems for nonleft-invertible systems is more elegant since it does not require any preliminary algebra, it is computationally more involved. The reason is that the algorithm for computing the state feedback matrix, which turns a controlled invariant into an invariant, is more complex if the assignment of internal eigenvalues is required, [19]. Thus, from an algorithmic point of view, the approach given in Remark 2 should be preferred, since it performs the eigenvalue assignment on the n-dimensional triple (A, B, C) thus simplifying the basis transformation for the $2n$-dimensional Hamiltonian system.

VI. EXTENSION TO SINGULAR AND REGULAR LQR PROBLEMS

Singular and regular LQR optimal control problems are those where the control-input weighting matrix is positive–semidefinite (but not zero) and positive–definite, respectively. In this section, it will be shown that singular and regular problems can be stated in a cheap control framework for suitably extended systems.

Let us consider Problem 1 with (1) replaced by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \quad x(0) = x_0, \quad k = 0, 1, \ldots \\ y(k) &= Cx(k) + Du(k). \end{aligned} \tag{26}$$

Recast the optimal control problem in a cheap control setting as follows. Consider Problem 1 for the auxiliary extended system $(k = -1, 0, \ldots)$

$$\begin{aligned} \begin{bmatrix} x_C(k+1) \\ x_D(k+1) \end{bmatrix} &= \begin{bmatrix} A & B \\ 0 & -I_p \end{bmatrix} \begin{bmatrix} x_C(k) \\ x_D(k) \end{bmatrix} + \begin{bmatrix} 0 \\ I_p \end{bmatrix} v_C(k) \\ y_C(k) &= [C \quad D] \begin{bmatrix} x_C(k) \\ x_D(k) \end{bmatrix} \end{aligned} \tag{27}$$

with initial state

$$\begin{bmatrix} x_C(-1) \\ x_D(-1) \end{bmatrix} := \begin{bmatrix} A^{-1}x_0 - A^{-1}B\bar{u} \\ \bar{u} \end{bmatrix} \tag{28}$$

where $\bar{u} \in \mathbb{R}^p$ is arbitrary, and cost function

$$J_C := \sum_{k=-1}^{\infty} y_C(k)^T y_C(k). \tag{29}$$

Compare the original system (26) and the extended system (27). Controllability, left invertibility and absence of invariant zeros in $\mathbb{C}^\circ$ for (26) imply the same properties for (27). In fact, (27) can be obtained from (26) by inserting a delay with unit feedback on the control input signal flow, and, as is well-known, inserting a delay does not add invariant zeros and does not modify controllability and left invertibility properties.

Remark 4: Compare Problem 1 for system (26), and the auxiliary cheap control problem defined by (27), (28), (29). With the given initial conditions, the restrictions to $k = 0, 1, \ldots$ of the trajectories $x_C(k)$ and $x_D(k)(k = -1, 0, \ldots)$ minimizing J_C respectively coincide with the trajectories $x(k)$ and $u(k)(k = 0, 1, \ldots)$ minimizing J. Hence, the optimal control sequence $u(k)(k = 0, 1, \ldots)$ is retrieved from the optimal control sequence $v_C(k)(k = -1, 0, \ldots)$ by the recursive relationship $u(k+1) = -u(k) + v_C(k)(k = -1, 0, \ldots)$ with initial condition $u(-1) = \bar{u}$.

In fact, the initial condition (28) implies $x_C(0) = x_0$ for any $\bar{u} \in \mathbb{R}^p$. Furthermore, by construction, $x_D(k+1) = -x_D(k) + v_C(k)(k = -1, 0, \ldots)$.

Theorem 3: Let $[K_C \quad K_D]$ be the solution of the optimal control problem defined by (27), (28), (29). The solution of Problem 1 for system (26) is given by $K = K_C A^{-1}$.

Proof: According to the above remark, the equality

$$\begin{aligned} u(0) &= -u(-1) + v_C(-1) \\ &= -\bar{u} + K_C A^{-1} x_0 + (K_D - K_C A^{-1} B)\bar{u} \\ &= K_C A^{-1} x_0 + (K_D - K_C A^{-1} B - I_p)\bar{u} \end{aligned}$$

referring to the first transition, holds for all $\bar{u} \in \mathbb{R}^p$. Since $x(0) = x_0$ is fixed and $u(0)$ depends solely on $x(0)$, it follows that:

$$K_D - K_C A^{-1} B - I_p = 0. \tag{30}$$

Since (30) states a structural property of the extended system state feedback matrix, the thesis is proven. ■

VII. Conclusion

The discrete-time cheap control problem has been discussed in a geometric framework. A new solution to cheap, singular and regular LQR problems, based on the analysis of the geometric structure of the Hamiltonian system, has been proposed. The optimal state feedback matrix has been derived by recasting the cheap control problem as a decoupling problem and using the basic tools of the geometric approach. The geometric algorithm is not iterative and, consequently, exhibits a running time independent of numerical data.

Solving the discrete-time cheap control problem in a complete geometric framework provides some useful insight on the optimal control way of working. It clearly appears that the number of initial dead-beat steps is equal to the number of steps necessary to build the conditioned invariants $\hat{\mathcal{S}}_1^*$ or $\hat{\mathcal{S}}^*$ of the Hamiltonian system, while the exponential modes converging to zero corresponds to the internal eigenvalues of the controlled invariant $\hat{\mathcal{V}}^* \cap \hat{\mathcal{V}}_1^*$.

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[2]Available at www.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm

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