PERGAMON

Automatica 38 (2002) 1695–1703

automatica

www.elsevier.com/locate/automatica

Brief Paper

Convolution profiles for right inversion of multivariable non-minimum phase discrete-time systems[☆]

Giovanni Marro[a,*], Domenico Prattichizzo[b], Elena Zattoni[a]

[a] *DEIS, Università di Bologna, Viale Risorgimento, 2, 40136 Bologna, Italy*
[b] *DII, Università di Siena, Italy*

Received 23 October 2000; received in revised form 4 December 2001; accepted 29 March 2002

Abstract

The problem of the non-causal inversion of linear multivariable discrete-time systems is analyzed in the geometric approach framework and is solved through the computation of convolution profiles which guarantee perfect tracking under the assumption of infinite-length preaction and postaction time intervals. It is shown how the shape of the convolution profiles is related to both the relative degree and the invariant zeros of the plant. A computational setting for the convolution profiles is derived by means of the standard geometric approach tools. Feasibility constraints are also taken into account. A possible implementation scheme, based on a finite impulse response system acting on a stabilized control loop, is provided.

Keywords: Linear multivariable systems; Geometric approach; Non-minimum phase systems; Right inversion

1. Introduction

The problem of deriving a right or left inverse for a multivariable dynamical system has been widely studied in the past. Structural conditions for multivariable system invertibility were almost contemporarily derived in Dorato (1969), Sain and Massey (1969) and Silverman (1969). Equivalent structural conditions, expressed in geometric terms, were stated in Basile and Marro (1973). It is well known that right inversion is strictly connected to perfect tracking. In Francis (1979), perfect tracking was approached as an LQ cheap control problem, while in Davison and Scherzinger (1987) and Qiu and Davison (1993) it was shown that perfect tracking cannot be achieved if the system is non-minimum phase. However, during the last few years, it has been shown by several authors that the perfect tracking problem is solvable with bounded control effort if the signal to be tracked is previewed by a significant amount of time and, in the continuous-time case, if it is also sufficiently smooth. The first contributions to stable inversion of non-minimum phase SISO systems mainly refer to the continuous-time case and can be found in Devasia, Chen, and Paden (1996) and Hunt, Meyer, and Su (1996). The formalization of the so-called *steering along zeros technique* in the discrete-time SISO case is almost contemporary, see, e.g. Gross and Tomizuka (1994), Gross, Tomizuka, and Messner (1994), Tsao (1994), Marro and Fantoni (1996), and Marconi, Marro, and Melchiorri (2001). Significant recent contributions addressing the use of preview and preaction for stable inversion of discrete-time MIMO systems are given in Zou and Devasia (1999) and Zeng and Hunt (2000) for linear and nonlinear systems, respectively. The main contribution of the present work is the development of a novel solution, completely embedded in the geometric approach framework, to the non-causal inversion of discrete time, linear, non-minimum phase, right- and left-invertible systems in the multivariable case.

In order to clarify the significance of this paper with respect to the previous literature dealing with the SISO case, or, in other words, in order to explain in which sense the present paper extends some previous results valid for SISO systems, it must first be said that the computational algorithm providing the convolution profiles for perfect tracking presented in this paper is based on the evaluation of three

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by the Associate Editor Kenko Uchida under the direction of Editor Tamer Basar.

* Corresponding author. Tel.: +39-051-2093046; fax: +39-051-2093073.

E-mail addresses: gmarro@deis.unibo.it (G. Marro), prattichizzo@dii.unisi.it (D. Prattichizzo), ezattoni@deis.unibo.it (E. Zattoni).

0005-1098/02/$ - see front matter © 2002 Elsevier Science Ltd. All rights reserved.
PII: S0005-1098(02)00088-2

different control sequences: the infinite-horizon preaction associated with the unstable invariant zeros, the infinite-horizon postaction associated with the stable invariant zeros and the relative degree preaction. In the SISO case—see, e.g. Marconi et al. (2001)—there was no distinction between the relative-degree preaction and the infinite-horizon preaction; furthermore, both the infinite-horizon preaction and the infinite-horizon postaction were evaluated by strictly using the transfer function approach. Instead, in the MIMO case a pure geometrical approach has been adopted: the evaluation of the convolution profiles is based on the decomposition of the state space into three different invariant subspaces, respectively associated to the infinite-horizon preaction, the infinite-horizon postaction, and the relative degree preaction. In particular, the introduction of the concept of relative-degree preaction exploits an interesting property of the multivariable relative degree involving the minimum-conditioned invariant computational sequence.

Hence, the feature distinguishing this paper from other, valuable, works dealing with the multivariable case—e.g. the aforementioned (Zou & Devasia, 1999; Zeng & Hunt, 2000)—is that the role played by the geometric approach is crucial: the overall problem is solved in terms of controlled and conditioned invariant subspaces. Since the solution herein proposed is based on the decomposition of an invertible multivariable system, other powerful methods could also be used to effectively derive the convolution profiles, primarily the one presented in Sannuti and Saberi (1987)—see also Saberi, Sannuti, and Chen (1995) and Saberi, Stoorvogel, and Sannuti (2000)—where geometrical spaces are mapped to a special coordinate system. However, the classical geometric approach—Wonham (1985) and Basile and Marro (1992)—seems to be preferable within the scope of this paper because of its simplicity: being a *coordinate-free* approach, it provides immediate insight into the meaning of the procedures which are developed.

As for the paper content, first, it is recalled that perfect tracking can be achieved by the convolution of the signals to be tracked with suitable profiles defined on the whole time axis, from minus infinity to plus infinity. In this context, the convolution profiles are fluency derived from the fundamental properties of the controlled and conditioned invariant subspaces. Then, feasibility constraints are taken into account and it is shown that almost perfect tracking can be achieved by the convolution of a receding-horizon finite-number of samples of the signals to be tracked with consistently truncated profiles. The solution proposed herein is fully constructive and is supported by computational algorithms where only the very basic tools of the geometric approach, i.e. the Matlab© subroutines for geometric approach computations first published with the aforementioned (Basile and Marro, 1992), are employed.

Interesting problems connected with real implementation, mainly the effects of plant uncertainties, have been intentionally left apart from considerations, since the aim of this paper is just to outline—in the simplest way—a neat computational procedure to solve the non-causal inversion problem with stability in the MIMO case. However, the robustness of the implementation scheme depicted herein could be improved by taking appropriate precautions, first of all by decreasing sensitivity in a suitable frequency interval by an inner feedback. More generally, plant uncertainties could be faced by resorting to mixed geometrical-optimization procedures to design convolution profiles based on different criteria, i.e. optimal or suboptimal tracking with respect to some, suitably chosen, norm.

The paper is organized as follows. In Section 2, the notation and some preliminary results are introduced, directly starting from right- and left-invertible square systems, to which the results presented in the rest of the paper mainly apply. In Section 3, the basic notions of preaction and postaction are defined and their connections with structural perfect tracking and perfect tracking with internal stability are analyzed. The role played by relative degree and invariant zeros is a key point of this analysis. In Section 4 a possible implementation scheme for achieving almost perfect tracking is provided. The theory is illustrated by means of a numerical example in Section 5. Finally, in Section 6 the primary results are summarized and directions for future research are outlined.

2. Notation and recalls

Throughout this paper, $\mathbb{R}$ stands for the field of real numbers. Sets, vector spaces and subspaces are denoted by script capitals like $\mathscr{V}$, matrices and linear maps by slanted capitals like A, the image and the null space of A by $\operatorname{im} A$ and $\ker A$, respectively, the pseudo-inverse of A by $A^{\#}$, and the spectrum of A by $\sigma(A)$. The three-map discrete-time-invariant and linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

is considered, with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^p$ and controlled output $y \in \mathbb{R}^q$, matrices B and C of full rank. Referring to system (1), $\mathscr{B}$ stands for $\operatorname{im} B$, $\mathscr{C}$ for $\ker C$, $\mathscr{V}^*$ or $\max \mathscr{V}(A, \mathscr{B}, \mathscr{C})$ for the maximum $(A, \mathscr{B})$-controlled invariant contained in $\mathscr{C}$, $\mathscr{S}^*$ or $\min \mathscr{S}(A, \mathscr{C}, \mathscr{B})$ for the minimum $(A, \mathscr{C})$-conditioned invariant containing $\mathscr{B}$, and $\mathscr{R}_{\mathscr{V}^*}$ for the reachable set on $\mathscr{V}^*$, computable as $\mathscr{V}^* \cap \mathscr{S}^*$. From now on, system (1) is assumed to be square, right- and left-invertible.

The system right invertibility (or functional controllability) denotes its property of reproducing at the output any arbitrary function, starting from the zero state and after some delay, provided that a suitable control input is applied. The relative degree of a (right-invertible) system denotes the minimum delay for any output to be reproduced. Clearly, the right invertibility assumption is mandatory to guarantee solvability of the perfect tracking problem. On the contrary, the left invertibility assumption is introduced to simplify the discussion, which, nevertheless, could be extended to non-left-invertible systems with the difference that the eigenvalues associated with $\mathscr{R}_{\mathscr{V}^*}$ should be arbitrarily assigned by feedback—see, e.g. Remark 2 in Marro, Prattichizzo, and Zattoni (2002) for details. In the following, some characterization of right and left invertibility, the connection between the relative degree and the standard conditioned invariant algorithm, and, finally, the definition of invariant zeros are briefly recalled, since they are functional to the main results on perfect tracking presented in Section 3.

Proposition 1. *Functional controllability of system* (1) *is equivalent to each one of the following relations*:

$$C\mathscr{S}^* = \mathbb{R}^q, \tag{2}$$

$$\mathscr{S}^* + \mathscr{C} = \mathbb{R}^n, \tag{3}$$

$$\mathscr{S}^* + \mathscr{V}^* = \mathbb{R}^n. \tag{4}$$

Proof. Condition (2) is simply the discrete-time counterpart of a result proved by Basile and Marro (1973, Theorem 4). Since C is of full rank, (2) and (3) are equivalent (recall that $C\mathscr{S}^* = CP\mathscr{S}^*$, where P denotes the orthogonal projection matrix on $\operatorname{im} C^{\mathrm{T}}$, i.e. the projection matrix on $\operatorname{im} C^{\mathrm{T}}$ along $\ker C$). Equivalence of (3) and (4) follows as the dual of the well-known property

$$\mathscr{V}^* \cap \mathscr{B} = \{0\} \quad \Leftrightarrow \quad \mathscr{R}_{\mathscr{V}^*} = \mathscr{V}^* \cap \mathscr{S}^* = \{0\}, \tag{5}$$

where the expression for $\mathscr{R}_{\mathscr{V}^*}$ was first proved by Morse (1973, Lemma 1.1.) □

Proposition 2. *Left invertibility of system* (1) *is equivalent to each one of relations* (5). *Hence, right and left invertibility of system* (1) *is equivalent to the relation* $\mathscr{V}^* \oplus \mathscr{S}^* = \mathbb{R}^n$.

Theorem 3. *The relative degree of system* (1)*—right-invertible by assumption—is the least integer* ρ *such that each one of the following relations holds*:

$$C\mathscr{S}_{\rho-1} = \mathbb{R}^q, \tag{6}$$

$$\mathscr{S}_{\rho-1} + \mathscr{C} = \mathbb{R}^n, \tag{7}$$

$$\mathscr{S}_{\rho-1} + \mathscr{V}^* = \mathbb{R}^n, \tag{8}$$

where $\mathscr{S}_i$, $i = 1, 2, \ldots, k$, *is provided by the standard conditioned invariant algorithm*

$$\mathscr{S}_0 := \mathscr{B}, \quad \mathscr{S}_i := A(\mathscr{S}_{i-1} \cap \mathscr{C}) + \mathscr{B}, \ \ i = 1, 2, \ldots, k \tag{9}$$

with k *such that* $\mathscr{S}_{k+1} = \mathscr{S}_k$.

Proof. Eq. (6) is an obvious consequence of the definition of right invertibility and Eq. (2). Equivalence of (6) and (7) is straightforward. Equivalence of (7) and (8) can be shown by slightly extending the proof of the Morse Theorem. Consider the standard controlled invariant sequence

$$\mathscr{V}_0 := \mathscr{C}, \quad \mathscr{V}_i := A^{-1}(\mathscr{V}_{i-1} + \mathscr{B}) \cap \mathscr{C}, \ \ i = 1, 2, \ldots, k$$

with k such that $\mathscr{V}_{k+1} = \mathscr{V}_k$, and the modified sequence

$$\mathscr{V}'_0 := \mathscr{C}, \quad \mathscr{V}'_i := A^{-1}(\mathscr{V}'_{i-1} + \mathscr{S}_{\rho-1}) \cap \mathscr{C}, \ \ i = 1, 2, \ldots, k.$$

As can be easily shown by induction, the relations

$$\mathscr{S}_{\rho-1} + \mathscr{V}_i = \mathscr{S}_{\rho-1} + \mathscr{V}'_i, \quad i = 0, 1, \ldots, k$$

hold. Hence, in particular, it is possible to write $\mathscr{S}_{\rho-1} + \mathscr{V}^* = \mathscr{S}_{\rho-1} + \mathscr{V}'^*$, where $\mathscr{V}^*$ and $\mathscr{V}'^*$ denote the last terms of the respective sequences. However, by construction, $\mathscr{V}'_i = \mathscr{C}$, $i = 1, 2, \ldots, k$, can be written. Hence the thesis is proved. □

Corollary 4. *The relative degree* ρ *of system* (1)*—right- and left-invertible by assumption—is the number of steps for evaluating* $\mathscr{S}^*$, *i.e.* $\mathscr{S}_{\rho-1} = \mathscr{S}^*$.

Definition 5. The invariant zeros of system (1) are the internal unassignable eigenvalues of $\mathscr{V}^*$, defined by

$$\mathscr{Z} = \sigma(A + BF)_{\mathscr{V}^*/\mathscr{R}_{\mathscr{V}^*}},$$

where F denotes any matrix such that $(A + BF)\mathscr{V}^* \subseteq \mathscr{V}^*$.

Since system (1) is left-invertible by assumption, all the internal eigenvalues of $\mathscr{V}^*$ are unassignable.

3. Basic results

If a system is right-invertible, then, by definition, arbitrary trajectories can be reproduced at the outputs provided that they are known in advance by ρ instants of time, where ρ denotes the system relative degree. Hence, perfect tracking per se only requires a preview of ρ samples of the reference signals. However, if the system is non-minimum phase, both the states and the control inputs may diverge exponentially. Clearly, this is not acceptable in technical practice and, if a significant preview of the reference signals is available, it can be avoided by means of a *preaction*, i.e. a suitable action in advance on the control inputs. The problem of reproducing a given output without caring about state divergence is also known as the *structural* perfect tracking problem, and is actually the problem which was considered in the early investigations on system invertibility. Instead, the problem of perfect tracking *with internal stability* consists in reproducing a given output at a certain time instant, while maintaining the output equal to zero elsewhere and avoiding state divergence.

The solution which is proposed in this section applies to the more general problem, the one with internal stability, and is achieved by combining (Corollary 13) results on structural perfect tracking (Lemma 6, Algorithm 7, and Theorem 9) with results on internal stability (Theorem 10 and Algorithm

12). More specifically, Theorem 9 shows that the state can be driven from $x(0)=0$ to $x(\rho)=x_f$ along a trajectory which is invisible at output until the step $\rho-1$, while $Cx(\rho)$ is equal to the desired output y_f. Theorem 9 essentially exploits the geometric meaning of the conditioned invariant subspace $\mathscr{S}^*$. Theorem 10 shows that the effect at the output of a generic state x_f can be nulled while avoiding state divergence. Theorem 10 contains the main design idea in the sense that it introduces the concepts of preaction and postaction. However, since the target is to reproduce at the output any given sequence (with a possible relative degree delay), both Theorems 10 and 9 are needed: Theorem 9 shows how any given output can be obtained at the step ρ, while Theorem 10 shows how the effect at the output due to the state which guarantees the desired output at the step ρ can be nulled from the step $\rho+1$ on, still ensuring a state trajectory converging to the origin. This concept is expressed by Corollary 13.

In order to formulate the following statements concisely, some further assumptions on the controlled system are introduced, all of them of a technical nature: the system is also assumed to be controllable, asymptotically stable, [1] with no invariant zeros on the unit circle.

Lemma 6. *Let us assume that A is non-singular and denote with (A_r, B_r) the pair characterizing the reverse dynamics of system* (1), *i.e. $A_r := A^{-1}$, $B_r := -A^{-1}B$. The subspace $\mathscr{S}^* \cap \mathscr{C}$ is an $(A_r, \mathscr{B}_r)$-controlled invariant and all its internal unassignable eigenvalues are equal to zero.*

Proof. The $(A_r, \mathscr{B}_r)$-controlled invariance of $\mathscr{S}^* \cap \mathscr{C}$ is proved by

$$\begin{aligned} A_r(\mathscr{S}^* \cap \mathscr{C}) &\subseteq A_r\mathscr{S}^* \cap A_r\mathscr{C} \\ &= A_r(A(\mathscr{S}^* \cap \mathscr{C}) + \mathscr{B}) \cap A_r\mathscr{C} \\ &= (\mathscr{S}^* \cap \mathscr{C} + A^{-1}\mathscr{B}) \cap A^{-1}\mathscr{C} \\ &\subseteq \mathscr{S}^* \cap \mathscr{C} + \mathscr{B}_r. \end{aligned}$$

For any given state $x_f \in (\mathscr{S}^* \cap \mathscr{C})$, one and only one trajectory belonging to $\mathscr{S}^* \cap \mathscr{C}$ exists, along which the state is driven from the origin to x_f. In fact, if there were two, their difference, still belonging to $\mathscr{S}^* \cap \mathscr{C}$, would lead the state back to the origin and this is against the hypothesis of left invertibility of the system. The same trajectory, followed backwards according to the reverse system dynamics, leads the state from x_f to the origin. This implies that all the internal eigenvalues of $\mathscr{S}^* \cap \mathscr{C}$ as an $(A_r, \mathscr{B}_r)$-controlled invariant are equal to zero. By the left-invertibility assumption, these eigenvalues are also unassignable. □

Algorithm 7 (Computation of the control sequence $u(k)$, $k=0,\dots,\rho$-2, which drives the state from $x(0)=0$ to $x(\rho$-1$) = \bar{x} \in \mathscr{S}^* \cap \mathscr{C}$, along a trajectory belonging to $\mathscr{S}^* \cap \mathscr{C}$). *Let $\bar{x}$ be any state belonging to $\mathscr{S}^* \cap \mathscr{C}$. Eqs.* (2) *and* (9) *imply that $\bar{x}$ can be reached from the origin in $\rho-1$ steps. By virtue of Lemma* 6, *the trajectory along which the state is driven from $x(\rho-1)=\bar{x}$ to the origin according to the dynamics of the closed-loop reverse system is given by*

$$x(k-1) = (A_r + B_rF_r)x(k), \quad k=\rho-1,\dots,1,$$

where F_r is such that $\mathscr{S}^* \cap \mathscr{C}$ is an invariant in $A_r + B_rF_r$. The corresponding control input sequence is given by

$$u(k) = F_r x(k+1), \quad k=0,\dots,\rho-2.$$

Remark 8. Algorithm 7 can also be used if A is singular, by resorting to a simple contrivance. In this case, a suitable pole placement can be performed, since (A,B) is controllable. Let H be such that $\bar{A} := A + BH$ is non-singular and denote by $\bar{u}(k)$ the control sequence obtained for the triple $(\bar{A}, B, C)$. Since the algorithm also provides the state $x(k)$, the control for (A,B,C) is given by

$$u(k) = \bar{u}(k) + Hx(k), \quad k=0,\dots,\rho-2.$$

Theorem 9. *For any given output $y_f \in \mathbb{R}^q$, a control input sequence $u(k)$, $k=0,\dots,\rho-1$, exists, which drives the state from $x(0)=0$ to $x(\rho)=x_f$, where x_f is such that $Cx_f=y_f$, along a trajectory belonging to $\mathscr{S}^* \cap \mathscr{C}$* (*therefore invisible at the output*) *until the last step but one.*

Proof. The right invertibility of the system implies Eq. (2). Therefore, for any given $y_f \in \mathbb{R}^q$, $x_f \in \mathscr{S}^*$ exists such that $Cx_f = y_f$. Since $\mathscr{S}^* = A(\mathscr{S}^* \cap \mathscr{C}) + \mathscr{B}$, $\bar{x} \in (\mathscr{S}^* \cap \mathscr{C})$ and $\mu \in \mathbb{R}^q$ exist such that $x_f = A\bar{x} + B\mu$. Algorithm 7 provides the control input sequence that drives the state from the origin to $\bar{x}$ along a trajectory belonging to $\mathscr{S}^* \cap \mathscr{C}$, while the control input that drives the state from $\bar{x}$ to x_f is $u(\rho-1)=\mu$. Let V_r be a basis matrix of $\mathscr{S}^* \cap \mathscr{C}$, so that $\bar{x} = V_r\beta$. The relation

$$\begin{bmatrix} \beta \\ \mu \end{bmatrix} = (C \;\; [AV_r \;\; B])^{\#} y_f \tag{10}$$

provides β and μ. □

Theorem 10. *Let us assume that the state of system* (1) *is forced to a given $x_f \in \mathbb{R}^n$ at the time instant ρ by an external event. A control input sequence $u(k)$, $k=\dots,0,1,\dots$, exists which both nulls the effect of the state x_f on the output and avoids state divergence.*

Proof. By the assumption of right- and left invertibility of the system, x_f can be decomposed as $x_f = \bar{x}_{\mathscr{V}^*} + \bar{x}_{\mathscr{S}^*}$, where $\bar{x}_{\mathscr{V}^*} \in \mathscr{V}^*$ and $\bar{x}_{\mathscr{S}^*} \in \mathscr{S}^*$. In the proof of Theorem 9 it has been shown that a control input sequence, $u_1(k)$, $k=0,\dots,\rho-1$, exists which drives the state from $x(0)=0$ to $x(\rho) = -\bar{x}_{\mathscr{S}^*}$, thus cancelling the effect of $\bar{x}_{\mathscr{S}^*}$ on the output from the time instant ρ on. The component $\bar{x}_{\mathscr{V}^*}$ is

[1] Stability is often ensured by feedback—see Section 4.

forced to remain on $\mathscr{V}^*$ and to asymptotically converge to the origin by means of the infinite-horizon preaction and postaction control sequences. The left-invertibility assumption implies that the equality $\mathscr{V}^* = \mathscr{V}_S \oplus \mathscr{V}_U$ holds, where $\mathscr{V}_S$ and $\mathscr{V}_U$ are $(A, \mathscr{B})$-controlled invariants, strictly stable and strictly unstable, respectively. Then $\bar{x}_{\mathscr{V}^*}$ can be decomposed as $\bar{x}_{\mathscr{V}^*} = \bar{x}_{\mathscr{V}_S} + \bar{x}_{\mathscr{V}_U}$, where $\bar{x}_{\mathscr{V}_S} \in \mathscr{V}_S$ and $\bar{x}_{\mathscr{V}_U} \in \mathscr{V}_U$. Let F be such that $\mathscr{V}^*$, $\mathscr{V}_S$ and $\mathscr{V}_U$ are $A + BF$ invariants. Since all the internal eigenvalues of $\mathscr{V}_S$ in $A + BF$ are stable, a control input sequence, $u_2(k)$, $k = \rho, \rho + 1, \ldots$, exists which drives $\bar{x}_{\mathscr{V}_S}$ asymptotically to the origin, along a trajectory belonging to $\mathscr{V}_S$. Since all the internal eigenvalues of $\mathscr{V}_U$ in $A + BF$ are unstable, a state trajectory belonging to $\mathscr{V}_U$ exists, which, pursued backwards in time, steers $-\bar{x}_{\mathscr{V}_U}$ asymptotically to the origin as k approaches $-\infty$. The corresponding control input sequence, denoted by $u_3(k)$, $k = \rho - 1, \rho - 2, \ldots$, is supposed to be recorded so that it can be applied, forwards in time, to the original system, starting from the zero state at the time instant $-\infty$. So, the control law $u_3(k)$, $k = \ldots, \rho - 2, \rho - 1$, applied to the original system, starting from the zero state at the time $-\infty$, leads the state to $-\bar{x}_{\mathscr{V}_U}$ at the time instant ρ, along an externally stable state trajectory. This means that it becomes possible to cancel $\bar{x}_{\mathscr{V}_U}$ at the same time instant ρ. In the most general case, the target specified in the statement is achieved by applying the sum of the previously defined control sequences (each assumed equal to zero wherever not explicitly defined). □

Remark 11. The above proof of Theorem 10 points out that, if the system has no invariant zeros, then only the relative-degree preaction $u_1(k)$, $k = 0, \ldots, \rho - 1$, has to be applied, if the system has only stable invariant zeros, then also the infinite-horizon postaction $u_2(k)$, $k = \rho, \rho + 1, \ldots$, has to be taken into account, and, finally, if the system has both stable and unstable invariant zeros, then the infinite-horizon preaction $u_3(k)$, $k = \ldots, \rho - 2, \rho - 1$, also has to be considered.

Algorithm 12 (Computation of the control sequences $u_1(k)$, $u_2(k)$, and $u_3(k)$). *Let V and V_r be basis matrices of $\mathscr{V}^*$ and $\mathscr{S}^* \cap \mathscr{C}$, respectively. Then $x_f \in \mathbb{R}^n$ can be decomposed as $x_f = V\alpha + AV_r\beta + B\mu$ where $\alpha \in \mathbb{R}^s$ with $s := \dim \mathscr{V}^*$, $\beta \in \mathbb{R}^t$ with $t := \dim(\mathscr{S}^* \cap \mathscr{C})$ and $\mu \in \mathbb{R}^q$ are given by*

$$\begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix} = [V \;\; AV_r \;\; B]^{\#} x_f. \tag{11}$$

The component $\bar{x}_{\mathscr{S}^*} = AV_r\beta + B\mu$ can be cancelled by reaching its opposite as specified in Theorem 9. The relative-degree control sequence $u_1(k)$ is so obtained. The component $\bar{x}_{\mathscr{V}^*}$ can be managed by applying the control input sequences resulting from the procedure described below. Let us perform the state space basis transformation $T := [V \;\; S]$, where S is a basis matrix of $\mathscr{S}^*$. The matrices A', B', C' in the new basis have the following structures:

$$A' := T^{-1}AT = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix}, \quad B' := T^{-1}B = \begin{bmatrix} 0 \\ B'_2 \end{bmatrix},$$

$$C' := CT = [0 \quad C'_2],$$

where $A'_{11} \in \mathbb{R}^{s \times s}$ and each of the other sub-matrices has accordingly defined dimensions. Let us consider a state feedback matrix $F' := [F'_1 \;\; 0]$, where $F'_1 := -(B'_2)^{\#} A'_{21}$. Then the closed-loop system matrix is

$$A'_F := A' + B'F' = \begin{bmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{bmatrix},$$

where $A'_{11} \in \mathbb{R}^{s \times s}$ is the restriction of A'_F to $\mathscr{V}^*$. By performing a further basis transformation $T' \in \mathbb{R}^{s \times s}$ separating the stable and unstable invariant subspaces of A'_{11} it follows that

$$A''_{11} := (T')^{-1} A'_{11} T' = \begin{bmatrix} A_S & 0 \\ 0 & A_U \end{bmatrix}.$$

The corresponding $F''_1 := T'F'_1$ can be accordingly partitioned as $F''_1 = [F_S \;\; F_U]$. Let

$$\begin{bmatrix} \bar{x}_S \\ \bar{x}_U \end{bmatrix} := (T')^{-1}\alpha,$$

the postaction state trajectory in the new basis is computed from the initial condition $x_2(\rho) = \bar{x}_S$ by the recursive formula

$$x_2(k+1) = A_S x_2(k), \quad k = \rho, \rho + 1, \ldots,$$

while the corresponding control input sequence is

$$u_2(k) = F_S x_2(k), \quad k = \rho, \rho + 1, \ldots .$$

The preaction state trajectory is similarly computed from the initial condition $x_3(\rho) = -\bar{x}_U$ by the recursive formula

$$x_3(k-1) = A_U^{-1} x_3(k), \quad k = \rho, \rho - 1, \ldots,$$

while

$$u_3(k) = F_U x_3(k), \quad k = \ldots, \rho - 2, \rho - 1$$

gives the corresponding control input sequence.

Corollary 13. *For any given $y_f \in \mathbb{R}^q$, a control input sequence reproducing y_f at the output at the time instant ρ exists, cancelling the output elsewhere and maintaining the state bounded.*

Proof. Such a sequence can be obtained by combining the following two: the one leading the state from $x(0) = 0$ to $x(\rho) = x_f$, where x_f is such that $Cx_f = y_f$, along a trajectory belonging to $\mathscr{S}^* \cap \mathscr{C}$ until the time instant $\rho - 1$ (Theorem 9) and the one cancelling the effect on the output of the state Ax_f that is produced at the time instant $\rho + 1$ by the former (Theorem 10). □

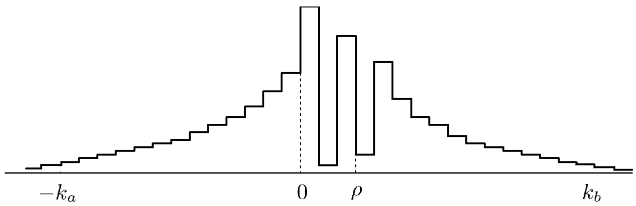

Fig. 1. A typical input component generating the impulse y_f at the time instant ρ.

The previous results focus on the problem of reproducing at the output an arbitrary reference vector $y_f \in \mathbb{R}^q$, with the relative-degree delay and the warranty of internal stability. In the most general case, the plot of a generic component of the control input as a function of time is a profile similar to that shown in Fig. 1, where preaction, relative-degree preaction, and postaction can be easily distinguished.

By virtue of superposition, particularly relevant to our aims is considering, in place of an arbitrary reference vector y_f, each component of some basis of the output space $\mathbb{R}^q$, for instance the main basis. The previously described algorithms still hold if the identity matrix I_q is assumed instead of y_f. Consistently, the state and the control input become matrices as well. More precisely, Eq. (10) is replaced by

$$\begin{bmatrix} \beta \\ \mu \end{bmatrix} = (C\ \ [AV_{\mathrm{r}}\ \ B])^{\#} I_q$$

and Eq. (11) is replaced by

$$\begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix} = [V\ \ AV_{\mathrm{r}}\ \ B]^{\#} X_f,$$

where $X_f := AV_{\mathrm{r}}\beta + B\mu$. As a consequence, the control sequence turns out to be a sequence of $q \times q$ matrices, denoted by $H(\ell)$, $\ell = \ldots, 0, 1, \ldots$. The generic element $H_{ij}(\ell)$, $\ell = \ldots, 0, 1, \ldots,$ is the control sequence to be applied at the jth input in order to obtain a unit impulse at the ith output at the time instant ρ. Obviously, in the most general case, its plot is still similar to that shown in Fig. 1. In conclusion, once the sequence $H(\ell)$, $\ell = \ldots, 0, 1, \ldots,$ has been obtained, superposition allows reproducing (with relative-degree delay and internal stability) any reference signal $r(k)$, $k = \ldots, 0, 1, \ldots,$ provided that it is completely known in advance. In fact, the corresponding control sequence is given by

$$u(k) = \sum_{\ell=-\infty}^{\infty} H(\ell) r(k-\ell). \tag{12}$$

4. The right inversion with a finite impulse response (FIR) system

In Section 3, the problem of the non-causal inversion of a multivariable discrete time-invariant system is completely and *exactly* solved from a theoretical viewpoint. However, severe limitations to the practical implementation of the proposed solution are found at a simple inspection of Eq. (12). The first, apparent, drawback is that the number of elements of the sequence of matrices $H(\ell)$, $\ell = \ldots, 0, 1, \ldots,$ which can be recorded in a digital processing unit is (obviously) finite. This is not a great problem, since, by virtue of the exponential convergence to zero of each profile towards both minus infinity and plus infinity, it is possible to consider (and register) only a finite number of elements of the sequence, with an approximation error which decreases as the number of elements which are considered increases. In addition, it is worth noting that, although in some cases the reference trajectories are completely known in advance (e.g. profiles to be tracked by machine tools), in many cases these are known in advance by a time interval of finite length (e.g. contact flight). Owing to this, we can assert that almost perfect tracking can be achieved if a preview of the reference trajectories is available, whose length is significantly greater than the maximum time constant associated to the inverses of the controlled system invariant zeros. If this is the case, the convolution in Eq. (12) can be replaced by

$$u(k) = \sum_{\ell=-k_a}^{k_{b-1}} H(\ell) r(k-\ell), \tag{13}$$

where $-k_a$ and k_b are the time instants where the convolution profiles are truncated towards minus infinity and plus infinity, respectively or, equivalently, k_a and k_b are the number of samples of preview and memory of the reference signals, respectively. In other words, Eq. (13) expresses the idea that, at any time instant k, the control input $u(k)$ is evaluated by the convolution of the same $k_a + k_b$ elements of the truncated sequence of matrices with $k_a + k_b$ samples ("centered" at the current time instant k) of the reference signals.

A possible implementation scheme is shown in Fig. 2. The controlled system is the three-map discrete time-invariant dynamical system (1), with the hypotheses introduced in Sections 2 and 3. The feedback regulator is designed to guarantee stability and good steady-state performance of the control loop, e.g. it can be based on the internal model principle. The feedforward unit is designed to guarantee almost perfect tracking. Its input is the previewed reference signal $r(k + k_a)$. Its outputs are the control input $u(k)$, evaluated according to Eq. (13), and the ρ-delayed reference signal $r(k - \rho)$, to be compared with the output of the control loop. Hence, the feedforward unit consists of a FIR system. Clearly, the effect of truncation at the output is a *tracking error* $e(k)$ which can be conveniently managed by the feedback regulator.

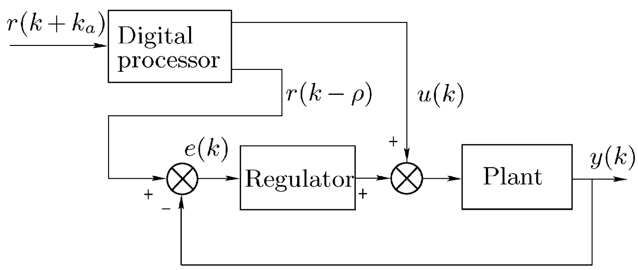

Fig. 2. The control system considered.

5. An illustrative example

The non-causal inversion of system (1) with

$$A := \begin{bmatrix} 0.5 & 1 & -0.4 & 0 \\ 0.1 & 0.7 & 0 & -0.5 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

is considered. The relative degree of the system is $\rho = 1$. Its invariant zeros are $z_1 = 0.8$ and $z_2 = 1.1$. The convolution profiles are computed with preaction and postaction time intervals consisting of 100 samples each.

The tracking of a reference signal, whose preview is assumed to correspond to 100 samples, is considered. The reference signal, shown in Fig. 3, is identically equal to zero at the first input, while it consists of the sequence of a ramp starting at the time instant 0, a constant and a negative step at the second input. Fig. 4 illustrates the outputs which are obtained by assuming the previously computed convolution profiles, i.e. exploiting the whole preview of the reference signal and a postaction of 100 samples. The ramp at the output starts with the relative degree delay, $\rho = 1$. It is also worth noting that the value of the second output is different from zero—although negligible with respect to the second—at the beginning of the preaction: this is caused by the truncation of the convolution profiles. A similar effect could also be seen at the end of the postaction, by a further visualization of some samples. Preaction and postaction can be easily distinguished in Fig. 5, which shows the control inputs. Preaction is particularly evident in the second control input before the time instant 0, while postaction is clear in the first control input after the time instant 40.

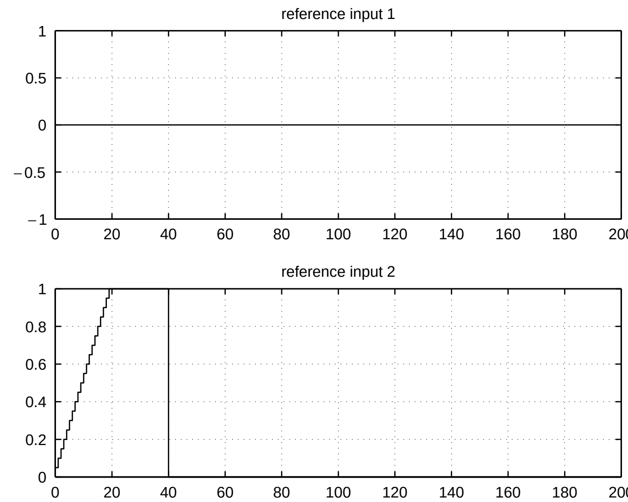

Fig. 3. Reference signals.

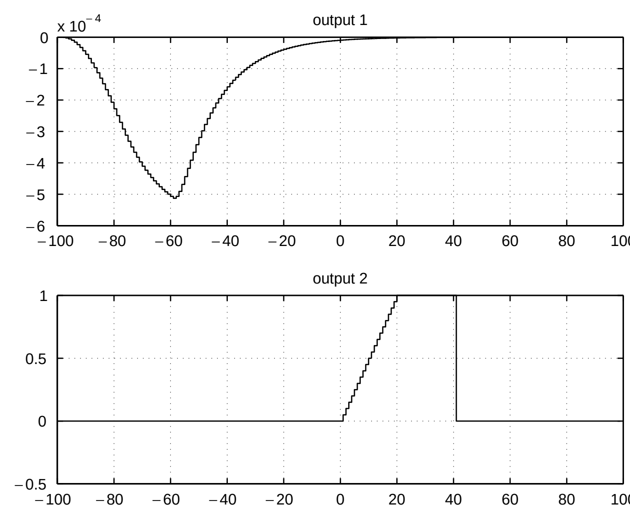

Fig. 4. Signals reproduced at the outputs.

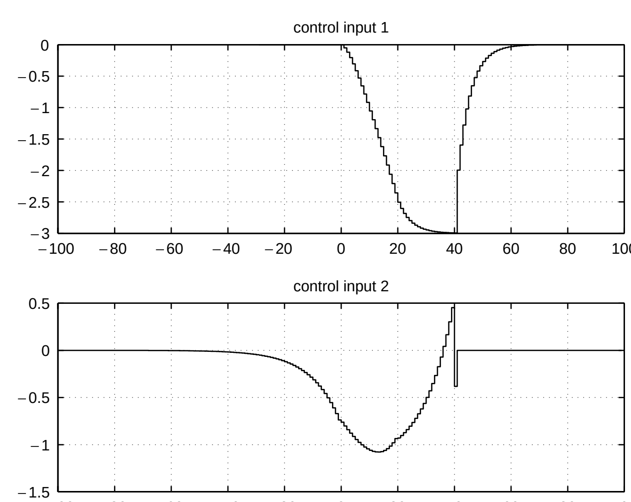

Fig. 5. Control inputs.

6. Concluding remarks

A complete computational setting for non-causal inversion of multivariable discrete time-invariant systems is provided in the geometric approach framework. The proposed solution is based on a FIR system evaluating the control inputs as a convolution of a receding-horizon finite number of samples of the signals to be tracked with suitable profiles. The connection between the shapes of the convolution profiles and the relative degree of the plant, as well as its invariant zeros is investigated in detail. The software supporting this work only exploits the very basic routines of the geometric approach. [2] The minimization of the effects of the truncation error, as well as the relation between the choice of the length of the preaction and postaction time intervals and the characteristics of the feedback regulator can be easily evaluated. Finally, it can be shown that, if the controlled system is both right- and left-invertible, similar profiles can be used for left inversion: a possible delay in the signal reconstruction replaces preaction.

References

Basile, G., & Marro, G. (1973). A new characterization of some structural properties of linear systems: Unknown-input observability, invertibility and functional controllability *International Journal of Control*, *17*(5), 931–943.

Basile, G., & Marro, G. (1992). *Controlled and conditioned invariants in linear system theory*. Englewood Cliffs, NJ: Prentice-Hall.

Davison, E. J., & Scherzinger, B. M. (1987). Perfect control of the robust servomechanism problem. *IEEE Transactions on Automatic Control*, *AC-32*(8), 689–701.

Devasia, S., Chen, D., & Paden, B. (1996). Nonlinear inversion-based output tracking. *IEEE Transactions on Automatic Control*, *41*(7), 930–942.

Dorato, P. (1969). On the inverse of linear dynamical systems. *IEEE Transactions on System Science and Cybernetics*, *SSC-5*(1), 43–48.

Francis, B. A. (1979). The optimal linear quadratic time invariant regulator with cheap control. *IEEE Transactions on Automatic Control*, *AC-24*, 616–621.

Gross, E., & Tomizuka, M. (1994). Experimental flexible beam tip tracking control with a truncated series approximation to uncancelable inverse dynamics. *IEEE Transactions on Control Systems and Technology*, *3*(4), 382–391.

Gross, E., Tomizuka, M., & Messner, W. (1994). Cancellation of discrete time unstable zeros by feedforward control. *ASME Journal of Dynamic Systems, Measurement and Control*, *116*(1), 33–38.

Hunt, L. R., Meyer, G., & Su, R. (1996). Non-causal inverses for linear systems. *IEEE Transactions on Automatic Control*, *41*(4), 608–611.

Marconi, L., Marro, G., & Melchiorri, C. (2001). A solution technique for almost perfect tracking of non-minimum-phase, discrete-time linear systems. *International Journal of Control*, *74*(5), 496–506.

Marro, G., & Fantoni, M. (1996). Using preaction with infinite or finite preview for perfect or almost perfect digital tracking. *In Proceedings of the Melecon '96—Eighth Mediterranean Electrotechnical Conference*, Vol. 1 (pp. 246–249). Bari, Italy.

Marro, G., Prattichizzo, D., & Zattoni, E. (2002). Geometric insight into discrete-time cheap and singular LQR problems. *IEEE Transactions on Automatic Control*, *47*(1), 102–107.

Morse, A. S. (1973). Structural invariants of linear multivariable systems. *SIAM Journal of Control*, *11*(3), 446–465.

Qiu, L., & Davison, E. J. (1993). Performance limitations of non-minimum phase systems in the servomechanism problem. *Automatica*, *29*(2), 337–349.

Saberi, A., Sannuti, P., & Chen, B. M. (1995). *H_2 optimal control*. System and Control Engineering. London: Prentice-Hall International.

Saberi, A., Stoorvogel, A. A., & Sannuti, P. (2000). *Control of linear systems with regulation and input constraints*. Communications and Control Engineering. Great Britain: Springer.

Sain, M. K., & Massey, J. L. (1969). Invertibility of linear time-invariant dynamical systems. *IEEE Transactions on Automatic Control*, *AC-14*(2), 141–149.

Sannuti, P., & Saberi, A. (1987). Special coordinate basis for multivariable linear systems—finite and infinite zero structure, squaring down and decoupling. *International Journal of Control*, *45*(5), 1655–1704.

Silverman, L. (1969). Inversion of multivariable linear systems. *IEEE Transactions on Automatic Control*, *AC-14*(3), 270–276.

Tsao, T.-C. (1994). Optimal feed-forward digital tracking controller design. *Journal of Dynamic Systems, Measurements and Control*, *116*, 583–592.

Wonham, W. M. (1985). *Linear multivariable control: A geometric approach* (3rd ed.). New York: Springer.

Zeng, G., & Hunt, L. R. (2000). Stable inversion for nonlinear discrete-time systems. *IEEE Transactions on Automatic Control*, *45*(6), 1216–1220.

Zou, Q., & Devasia, S. (1999). Preview-based stable-inversion for output tracking of linear systems. *Journal of Dynamic Systems*, *Measurement and Control*, *121*, 625–630, Also appeared in the *Proceedings of the American Control Conference*, Vol. 5 (pp. 3544–3548).

Giovanni Marro received the Laurea degree with honors in Electrical Engineering from the University of Bologna, Italy, in 1958. From 1970 he is Full Professor of Automatic Control at the University of Bologna. He has also been lecturer of a number of courses in the field of system theory at the Universities of Bari and Parma. He has spent some periods at several universities in the United States, among them: Case Institute of Technology, Cleveland, OH, University of California, Berkeley, and University of Florida, Gainesville. His research interests include the use of controlled invariants in control theory, dual lattice theory for approaching stabilizability, and optimization theory. He is the co-author, with G. Basile, of the well-known first paper of the geometric approach (JOTA, 1969). His contributions are documented by over a 100 of scientific papers and six books.

Domenico Prattichizzo received the M.S. degree in Electronics Engineering and the Ph.D. degree in Robotics and Automation from the University of Pisa, Pisa, Italy in 1991 and 1995, respectively. In 1994, he was a Visiting Scientist at the Artificial Intelligence Laboratory at Massachusetts Institute of Technology, Cambridge. Since 1991, he is a Research Associate in the Robotics and Automation group at the Centro "E. Piaggio" of the University of Pisa. From 1998 to 2001, he was Assistant Professor at the Department of Information Engineering of the University of Siena. Since 2002 he is Associate Professor at the University of Siena. Dr. Prattichizzo was co-organizer and co-chairman of sessions on control theory and robotics in many international conferences. His research activities include geometric

[2] An updated version is downloadable from the web site: http://www.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm.

control, optimal control, robotic manipulation, mobile robotics, and visual servoing. Since 2001, he serves as Member of the Editorial Board of the Journal of Dynamics of Continuous, Discrete and Impulsive Systems (DCDIS) Series B: Application and Algorithms and as Member of the Conference Editorial Board of the IEEE Control System Society. Dr. Prattichizzo is co-author of more than 80 papers in the area of control system and robotics.

Elena Zattoni received the Laurea degree with honors in Electronics Engineering and the Ph.D. degree in Systems Engineering from the University of Bologna, Italy, in 1995 and 1999, respectively. Currently she is with the Department of Electronics, Computer Science and Systems of the University of Bologna as Assistant Professor in Automatic Control Systems. Her thesis, on modeling and control of flexible mechanical structures, was awarded in 1997 by the Italian Society of Applied and Industrial Mathematics for the relevance of the subject to implementation. In 1998 she received a Brown University Exchange Program Fellowship Award, and in 1999 she was Visiting Scholar in the Laboratory for Engineering Man/Machine Systems at Brown University, Providence, RI. In 1998 and 1999 she was also lecturer of Advanced Automatic Control Systems at the University of Ferrara, Italy, and, since 2001, she is lecturer of Automatic Control Systems at the University of Bologna. Her research interests are focused on the geometric approach to control system theory and optimization theory, with special emphasis for the problems of non-causal inversion, signal decoupling and model matching with internal stability. She is author of several journal papers and papers in refereed conference proceedings. She serves as reviewer for Automatica, IEEE Transactions on Automatic Control and SIAM Journal on Control and Optimization. She is member of IFAC, IEEE CSS and SIMAI.