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Brief paper

Observer design via Immersion and Invariance for vision-based leader–follower formation control ☆

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ABSTRACT

The paper introduces a new vision-based range estimator for leader–follower formation control, based upon the Immersion and Invariance (I&I) methodology. The proposed reduced-order nonlinear observer is simple to implement, easy to tune and achieves global asymptotical convergence of the observation error to zero. Observability conditions for the leader–follower system are analytically derived by studying the singularity of the Extended Output Jacobian. The stability of the closed-loop system arising from the combination of the range estimator and an input-state feedback controller is proved by means of Lyapunov arguments. Simulation experiments illustrate the theory and show the effectiveness of the proposed designs.

1. Introduction

1.1. Motivation and related works

In the last few years we witnessed a growing interest in motion coordination and cooperative control of multi-agent systems. The research in this area has been stimulated by the recent technological advances in wireless communications and processing units, and by the observation that multiple agents can perform tasks far beyond the capabilities of a single robot (Bullo, Cortés, & Martínez, 2009). In this scenario, several new problems, such as, e.g. *consensus* (Olfati-Saber, Fax, & Murray, 2007), *rendezvous* (Lin, Morse, & Anderson, 2007), *coverage* (Cortés, Martínez, Karatas, & Bullo, 2004), *connectivity maintenance* (Dimarogonas & Kyriakopoulos, 2008) and *formation control*, have been introduced. Among these, due to its wide applicability domain, the formation control problem received a special attention and stimulated a great deal of research (Balch & Arkin, 1998; Das et al., 2002; Dong & Farrell, 2008; Tan & Lewis, 1997). By formation control we simply mean the problem of controlling the relative position and orientation of the robots in a group, while allowing the group to move as a whole. In a *leader–follower* formation control approach, the leader robot moves along a predefined trajectory while the other robots, the followers, are to maintain a desired distance and orientation to it (Das et al., 2002). Leader–follower architectures are known to have poor disturbance rejection properties. In addition, the over-reliance on a single agent for achieving the goal may be undesirable, especially in adverse conditions. Nonetheless, leader-following is particularly appreciated in the literature for its simplicity and scalability.

Recently, increasing attention has been devoted to sensing devices for autonomous navigation of multi-robot systems. A challenging and inexpensive way to address the navigation problem is to use exclusively on-board passive vision sensors, off-the-shelf cameras, which provide only the projection (or view-angle) to the other robots, but not the distance that must be estimated by a suitable observer.

A new observability condition for nonlinear systems based upon the Extended Output Jacobian, has been proposed in Mariottini, Pappas, Prattichizzo, and Daniilidis (2005) and applied to the study of formations localizability. The extended and unscented Kalman filters have been used in Mariottini et al. (2005) and Mariottini, Morbidi, Prattichizzo, Pappas, and Daniilidis (2007), respectively, to estimate the robots' relative distance (hereafter referred to as *range estimation*). Although widely used in the literature, these estimators are known to have some drawbacks: they are difficult to tune and implement, the estimation error is not guaranteed to converge to zero and an *a priori* knowledge about noise is usually required.

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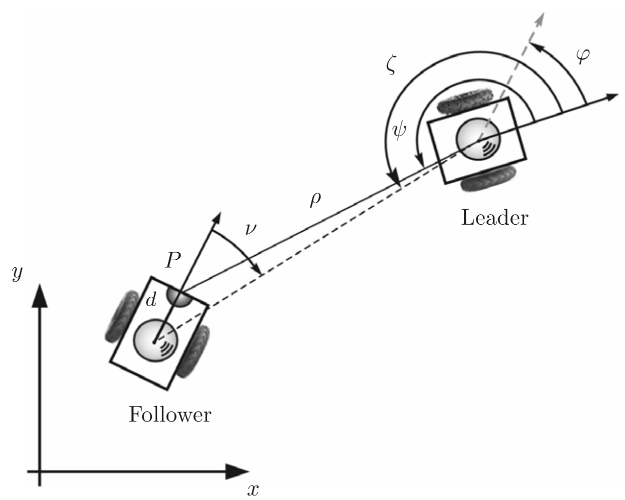

Fig. 1. Leader–follower setup.

A new methodology, called *Immersion and Invariance* (henceforth, I&I), has been recently proposed to design reduced-order observers for general nonlinear systems (Karagiannis, Carnevale, & Astolfi, 2008). In practice, the problem of designing a reduced-order observer is cast into the problem of rendering attractive an appropriately selected invariant manifold in the extended space of the plant and the observer. The effectiveness of the I&I observer design technique has been proved by Astolfi and coworkers through several academic and practical examples (see, e.g. Carnevale, Karagiannis, and Astolfi (2007), Carnevale and Astolfi (2008), Astolfi, Karagiannis, and Ortega (2008) and the references therein). However, to the best of our knowledge, this technique has never been applied in a formation control scenario.

1.2. Original contributions

The main focus of this paper is on observer design and nonlinear observability for vision-based leader–follower formations of unicycle robots. In particular, we present a nonlinear observer based upon the I&I methodology for leader–follower range estimation using on-board camera information (bearing-only). The reduced-order observer is globally asymptotically convergent, it is simple to implement and it can be easily tuned to achieve the desired convergence rate by acting on a single gain parameter. Second, we analytically derive an observability condition for the leader–follower system by studying the singularity of the Extended Output Jacobian. This condition, that naturally arises from the observer design procedure, is attractive since it allows one to identify those trajectories of the leader that preserve the observability of the system (see Mariottini et al. (2005) for more details). Finally, we use Lyapunov arguments to prove the stability of the closed-loop system arising from the combination of the range estimator and an input-state feedback control law.

1.3. Organization

The rest of the paper is organized as follows. Section 2 is devoted to the problem formulation. In Section 3 some basic facts about nonlinear observability are reviewed and the observability condition for the leader–follower system is introduced. In Section 4 an overview of the I&I observer design methodology is provided and in Section 5 the leader-to-follower range estimator is presented. In Section 6 the input-state feedback control law is introduced and the stability of the closed-loop system is proved via Lyapunov arguments. In Section 7 simulation experiments illustrate the theory and show the effectiveness of the proposed designs. Finally, in Section 8 the major contributions of the paper are summarized and possible avenues of future research are highlighted. A preliminary conference version of this paper appeared in Morbidi, Mariottini, and Prattichizzo (2008).

2. Problem formulation

The setup considered in the paper consists of two unicycle robots (see Fig. 1). One robot is the *leader*, whose control input is given by its translational and angular velocities, $[v_L \;\; \omega_L]^T$. The other robot is the *follower*,[1] controlled by $[v_F \;\; \omega_F]^T$. Each robot is equipped with an omnidirectional camera, which represents its only sensing device. Using well-known color detection and tracking algorithms (Forsyth & Ponce, 2002), the leader is able to measure both the angle ζ (w.r.t. the camera of the follower) and the angle ψ (w.r.t. a colored marker *P* placed at a distance *d* along the follower translational axis) (see Fig. 1). Analogously, the follower can compute the angle ν using its camera. We will assume that $\zeta, \psi, \nu \in [0, 2\pi)$.

Note that measuring ζ and ψ may not be a trivial task in practice, especially when the robots are distant. This problem has been addressed and solved in Mariottini et al. (2007), where only the angle ζ needs to be computed.

As shown in Das et al. (2002), the leader–follower kinematics can be fully described by polar coordinates, $[\rho \;\; \psi \;\; \varphi]^T$, where ρ is the distance from the leader to the marker *P* on the follower and φ is the relative orientation between the two robots, i.e. the bearing. It is easy to verify that,

$$\varphi = -\zeta + \nu + \pi. \tag{1}$$

Proposition 1 (Mariottini et al., 2005). *Consider the setup in Fig. 1. The leader–follower kinematics is described by the following driftless system,*

$$\begin{bmatrix} \dot{\rho} \\ \dot{\psi} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} \cos\gamma & d\sin\gamma & -\cos\psi & 0 \\ -\dfrac{\sin\gamma}{\rho} & \dfrac{d\cos\gamma}{\rho} & \dfrac{\sin\psi}{\rho} & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_F \\ \omega_F \\ v_L \\ \omega_L \end{bmatrix}, \tag{2}$$

where $\gamma \triangleq \varphi + \psi$. ■

The information flow between the leader and the follower is now briefly described. The follower transmits the angle ν to the leader and this robot computes the bearing φ using (1). To simplify the discussion, we will henceforth refer *only* to the bearing φ implicitly assuming the transmission of ν. The leader is then measuring a two dimensional vector,

$$y \triangleq [y_1 \;\; y_2]^T = [\psi \;\; \varphi]^T. \tag{3}$$

The leader will then use the image measurements in (3) to solve the range estimation problem by computing ρ (see Section 5).

3. Vision-based observability of the leader–follower system

3.1. Basics on nonlinear observability

In this section we briefly review some basic facts about the observability of nonlinear systems (Nijmeijer & van der Shaft, 1990).

[1] Note that the results of this paper can be easily extended to the case of multiple followers (Mariottini et al., 2005).

Consider a generic nonlinear system Σ of the form,

$$\Sigma : \begin{cases} \dot{s}(t) = f(s(t), u(t)), \quad s(0) = s_0, \\ y(t) \triangleq h(s(t)) = [h_1 \; h_2 \; \dots \; h_m]^T, \end{cases}$$

where $s(t) \in \mathcal{S}$ is the state, $y(t) \in \mathcal{Y}$ the observation vector and $u(t) \in \mathcal{U}$ the input vector. $\mathcal{S}$, $\mathcal{Y}$ and $\mathcal{U}$ are differential manifolds of dimension n, m and r, respectively.

The *observability problem* for Σ can be roughly viewed as the injectivity of the input–output map $\mathcal{R}_\Sigma : \mathcal{S} \times \mathcal{U} \mapsto \mathcal{Y}$ with respect to the initial conditions. Let $y(t, s_0, u) \triangleq h(s(t, s_0, u))$ denote the output of Σ at time t, for input u and initial state s_0. Two states $s_1, s_2 \in \mathcal{S}$ are said to be *indistinguishable* (denoted $s_1 I s_2$) for Σ if for every admissible input function u, the output function $y(t, s_1, u)$, $t \geq 0$, of the system for initial state s_1, and the output function $y(t, s_2, u)$, $t \geq 0$, of the system for initial state s_2, are identical on their common domain of definition.

The notions of observability and indistinguishability are tightly related, as shown in the following definition (Nijmeijer & van der Shaft, 1990):

Definition 1 (*Observability*). Given two initial states $s_1, s_2 \in \mathcal{S}$, the system Σ is called *observable*, if $s_1 I s_2$ implies $s_1 = s_2$.

A sufficient condition for the *local weak observability* of Σ has been introduced in Hermann and Krener (1977). An equivalent and more intuitive condition, based upon the Extended Output Jacobian, is presented in the next proposition (the proof is reported in Mariottini et al. (2005)).

Proposition 2 (*Observability Condition*). *System Σ is locally weakly observable at a point $s_0 \in \mathcal{S}$, if there exists an open neighborhood $\mathcal{D}$ of s_0 such that, for arbitrary $s \in \mathcal{D}$, the Extended Output Jacobian $J \in \mathbb{R}^{mn \times n}$ defined as the matrix with rows,*

$$J \triangleq \left\{ \frac{\partial}{\partial s} \left[\frac{\mathrm{d}^{(j-1)} h_i}{\mathrm{d}t^{(j-1)}} \right] \middle| \, i = 1, 2, \dots, m; j = 1, 2, \dots, n \right\},$$

is full rank. ■

Remark 1. Proposition 2 states that the observability of Σ can be tested by checking the rank of a matrix made of the state partial derivatives of the output vector and of all its $n-1$ time derivatives. In particular, it is evident that for Σ to be locally weakly observable, is sufficient that *at least one* $n \times n$ submatrix of J is full rank.

3.2. Observability of the leader–follower system

From Proposition 2 it turns out that system (2)–(3) is observable when at least one 3×3 submatrix of $J \in \mathbb{R}^{6\times 3}$ is nonsingular. Let us consider, e.g. the submatrix,

$$J^\star = \begin{bmatrix} \frac{\partial y_1}{\partial \rho} & \frac{\partial y_1}{\partial \psi} & \frac{\partial y_1}{\partial \varphi} \\ \frac{\partial \dot{y}_1}{\partial \rho} & \frac{\partial \dot{y}_1}{\partial \psi} & \frac{\partial \dot{y}_1}{\partial \varphi} \\ \frac{\partial y_2}{\partial \rho} & \frac{\partial y_2}{\partial \psi} & \frac{\partial y_2}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\partial \dot{\psi}}{\partial \rho} & \frac{\partial \dot{\psi}}{\partial \psi} & \frac{\partial \dot{\psi}}{\partial \varphi} \\ 0 & 0 & 1 \end{bmatrix}.$$

It is then obvious that if,

$$\det(J^\star) = \frac{1}{\rho^2} \left[-v_\mathrm{F} \sin\gamma + \omega_\mathrm{F} d \cos\gamma + v_\mathrm{L} \sin\psi \right] \neq 0, \tag{4}$$

the system is observable.

4. Observer design via I&I

For the reader's convenience we provide here a brief overview of the basic theory related to the observer design via I&I (Karagiannis et al., 2008). Consider a nonlinear, time-varying system described by,

$$\dot{y} = f_1(y, \eta, t), \tag{5}$$

$$\dot{\eta} = f_2(y, \eta, t), \tag{6}$$

where $y \in \mathbb{R}^m$ is the measured part of the state and $\eta \in \mathbb{R}^q$ is the unmeasured part of the state. The vector fields $f_1(\cdot), f_2(\cdot)$ are assumed to be forward complete, i.e. trajectories starting at time t^* are defined for all times $t \geq t^*$.

Definition 2. The dynamical system,

$$\dot{\xi} = \alpha(y, \xi, t), \tag{7}$$

with $\xi \in \mathbb{R}^p$, $p \geq q$, is called an *observer* for the system (5)–(6), if there exist mappings, $\beta(y, \xi, t) : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$ and $\phi_{y,t}(\eta) : \mathbb{R}^q \to \mathbb{R}^p$, with $\phi_{y,t}(\eta)$ parameterized by t and y and left-invertible, such that the manifold $\mathcal{M} = \{(y, \eta, \xi, t) \in \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R} : \beta(y, \xi, t) = \phi_{y,t}(\eta)\}$, has the following properties:

(1) All trajectories of the extended system (5)–(7) that start on the manifold $\mathcal{M}$ remain there for all future times, i.e. $\mathcal{M}$ is *positively invariant*.
(2) All trajectories of (5)–(7) that start in a neighborhood of $\mathcal{M}$ asymptotically converge to $\mathcal{M}$, i.e. $\mathcal{M}$ is *attractive*.

The above definition states that an asymptotic estimate $\hat{\eta}$ of η is given by $\phi^L_{y,t}(\beta(y, \xi, t))$, where $\phi^L_{y,t}$ denotes a left-inverse of $\phi_{y,t}$. The following proposition provides a general tool for constructing a nonlinear observer of the form given in Definition 2.

Proposition 3. *Consider the system (5)–(7) and suppose that there exist two mappings $\beta(\cdot) : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$ and $\phi_{y,t}(\cdot) : \mathbb{R}^q \to \mathbb{R}^p$, with a left-inverse $\phi^L_{y,t}(\cdot) : \mathbb{R}^p \to \mathbb{R}^q$, such that the following conditions hold:*

(A1) *For all y, ξ and t,* $\det\left(\frac{\partial \beta}{\partial \xi}\right) \neq 0$.

(A2) *The system,*

$$\begin{aligned} \dot{z} = & -\frac{\partial \beta}{\partial y}\left(f_1(y, \hat{\eta}, t) - f_1(y, \eta, t)\right) + \left.\frac{\partial \phi_{y,t}}{\partial y}\right|_{\eta=\hat{\eta}} f_1(y, \hat{\eta}, t) \\ & - \frac{\partial \phi_{y,t}}{\partial y} f_1(y, \eta, t) + \left.\frac{\partial \phi_{y,t}}{\partial \eta}\right|_{\eta=\hat{\eta}} f_2(y, \hat{\eta}, t) \\ & - \frac{\partial \phi_{y,t}}{\partial \eta} f_2(y, \eta, t) + \left.\frac{\partial \phi_{y,t}}{\partial t}\right|_{\eta=\hat{\eta}} - \frac{\partial \phi_{y,t}}{\partial t}, \end{aligned} \tag{8}$$

with $\hat{\eta} = \phi^L_{y,t}(\phi_{y,t}(\eta) + z)$, has an asymptotically stable equilibrium at $z = 0$, uniformly in η, y and t. Then system (7) with,

$$\begin{aligned} \alpha(y, \xi, t) = & -\left(\frac{\partial \beta}{\partial \xi}\right)^{-1} \left(\frac{\partial \beta}{\partial y} f_1(y, \hat{\eta}, t) + \frac{\partial \beta}{\partial t} \right. \\ & \left. - \left.\frac{\partial \phi_{y,t}}{\partial y}\right|_{\eta=\hat{\eta}} f_1(y, \hat{\eta}, t) - \left.\frac{\partial \phi_{y,t}}{\partial \eta}\right|_{\eta=\hat{\eta}} f_2(y, \hat{\eta}, t) - \left.\frac{\partial \phi_{y,t}}{\partial t}\right|_{\eta=\hat{\eta}} \right), \end{aligned}$$

where $\hat{\eta} = \phi^L_{y,t}(\beta(y, \xi, t))$, is a reduced-order observer for system (5)–(6). ■

Remark 2. Proposition 3 provides an implicit description of the observer dynamics (7) in terms of the mappings $\beta(\cdot)$, $\phi_{y,t}(\cdot)$ and $\phi^L_{y,t}(\cdot)$ which must then be selected to satisfy (A2). Hence, the problem of constructing a reduced-order observer for the system (5)–(6) reduces to the problem of rendering the system (8) asymptotically stable by assigning the functions $\beta(\cdot)$, $\phi_{y,t}(\cdot)$ and $\phi^L_{y,t}(\cdot)$. This peculiar stabilization problem can be extremely hard to solve, since, in general, it relies on the solution of a set of partial differential equations (or inequalities). However, as we will see in the next section, these equations are solvable in the problem under investigation.

5. Range estimator

In order to apply the procedure described in the Section 4 to design a nonlinear observer of the range ρ, system (2) should be recast in the form (5)–(6). To this end, it is convenient to introduce the new variable $\eta \triangleq 1/\rho$, that is well-defined assuming $\rho \neq 0$. System (2) then becomes,

$$\begin{bmatrix} \dot\eta \\ \dot\psi \\ \dot\varphi \end{bmatrix} = \begin{bmatrix} -\eta^2\cos\gamma & -\eta^2 d\sin\gamma & \eta^2\cos\psi & 0 \\ -\eta\sin\gamma & \eta d\cos\gamma & \eta\sin\psi & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_F \\ \omega_F \\ v_L \\ \omega_L \end{bmatrix}. \quad (9)$$

Recalling that $y \triangleq [\psi \;\; \varphi]^T$, system (9) can be rewritten as,

$$\begin{aligned} \dot y &= \underbrace{\begin{bmatrix} -\omega_L \\ \omega_L - \omega_F \end{bmatrix}}_{p(t)} + \underbrace{\begin{bmatrix} -v_F\sin\gamma + \omega_F d\cos\gamma + v_L\sin y_1 \\ 0 \end{bmatrix}}_{g(y,t)} \eta, \\ \dot\eta &= -\underbrace{(v_F\cos\gamma + \omega_F d\sin\gamma - v_L\cos y_1)}_{\ell(y,t)}\eta^2. \end{aligned} \quad (10)$$

The next proposition introduces a globally uniformly asymptotically convergent observer of η.

Proposition 4 (*Range Estimator*). *Suppose that the control inputs of the robots are bounded functions of time, i.e. $v_L, \omega_L, v_F, \omega_F \in \mathcal{L}^\infty$ and that v_L, v_F, ω_F are first order differentiable. Suppose that the following condition holds,*

$$|g_1(y,t)| \geq \mu > 0, \quad (11)$$

for some constant μ and for all t, where $g_1(y,t)$ is the first component of the vector $g(y,t)$ defined in (10). Then:

$$\begin{aligned} \dot\xi \;=\; & M[g_1(y,t), -v_F\sin\gamma + \omega_F d\cos\gamma](p(t) + g(y,t)\hat\eta) \\ & + M(\dot v_F\cos\gamma + \dot\omega_F d\sin\gamma - \dot v_L\cos y_1) \\ & - \frac{sign(g_1(y,t))}{g_1(y,t)^2}\,([\ell(y,t)\,,\, v_F\cos\gamma + \omega_F d\sin\gamma] \\ & \times\; (p(t) + g(y,t)\hat\eta)\hat\eta + (\dot v_F\sin\gamma - \dot\omega_F d\cos\gamma - \dot v_L\sin y_1)\hat\eta) \\ & + \frac{sign(g_1(y,t))}{g_1(y,t)}\ell(y,t)\hat\eta^2, \end{aligned} \quad (12)$$

is a globally uniformly asymptotically convergent observer for system (10), where M is a positive gain to be suitably tuned and,

$$\hat\eta = (M\ell(y,t) - \xi)|g_1(y,t)|. \quad (13)$$

Proof. With reference to the general design procedure presented in Section 4, let us suppose for simplicity, that $\phi_{y,t}(\eta) = \varepsilon(y,t)\eta$, where $\varepsilon(\cdot) \neq 0$ is a function to be determined (Carnevale et al., 2007). Consider an observer of the form given in Proposition 3,

$$\begin{aligned} \dot\xi \;=\; & -\left(\frac{\partial\beta}{\partial\xi}\right)^{-1}\left(\frac{\partial\beta}{\partial y}(p(t) + g(y,t)\hat\eta) + \frac{\partial\beta}{\partial t}\right. \\ & \left. - \frac{\partial\varepsilon}{\partial y}(p(t) + g(y,t)\hat\eta)\hat\eta - \frac{\partial\varepsilon}{\partial t}\hat\eta + \varepsilon(y,t)\ell(y,t)\hat\eta^2\right), \\ \hat\eta &= \varepsilon(y,t)^{-1}\beta(y,\xi,t). \end{aligned} \quad (14)$$

From (8) the dynamics of the error $z = \beta(y,\xi,t) - \varepsilon(y,t)\eta = \varepsilon(y,t)(\hat\eta - \eta)$ is given by,

$$\begin{aligned} \dot z = & -\left(\frac{\partial\beta}{\partial y}g(y,t) - \frac{\partial\varepsilon}{\partial y}p(t) - \frac{\partial\varepsilon}{\partial t}\right)\varepsilon(y,t)^{-1}z \\ & + \left(\frac{\partial\varepsilon}{\partial y}g(y,t) - \varepsilon(y,t)\ell(y,t)\right)(\hat\eta^2 - \eta^2). \end{aligned} \quad (15)$$

The observer design problem is now reduced to finding functions $\beta(\cdot)$ and $\varepsilon(\cdot) \neq 0$ that satisfy assumptions (A1)–(A2) of Proposition 3. In view of (15) this can be achieved by solving the partial differential equations,

$$\frac{\partial\beta}{\partial y}g(y,t) - \frac{\partial\varepsilon}{\partial y}p(t) - \frac{\partial\varepsilon}{\partial t} = \kappa(y,t)\varepsilon(y,t), \quad (16)$$

$$\frac{\partial\varepsilon}{\partial y}g(y,t) - \varepsilon(y,t)\ell(y,t) = 0, \quad (17)$$

for some $\kappa(\cdot) > 0$. From (17) we obtain the solution $\varepsilon(y,t) = -|g_1(y,t)|^{-1}$ which by (11) is well-defined and nonzero for all y and t. Let $\kappa(y,t) = M|g_1(y,t)|^3 + \left(\frac{\partial\varepsilon}{\partial y}p(t) + \frac{\partial\varepsilon}{\partial t}\right)|g_1(y,t)|$. By boundedness of the control inputs and $y(t)$, it is always possible to find $M > 0$ (sufficiently large) such that $\kappa(\cdot) > 0$. Eq. (16) is now reduced to $\frac{\partial\beta}{\partial y}g(y,t) = -Mg_1(y,t)^2$ which can be solved for $\beta(\cdot)$ yielding $\beta(y,\xi,t) = -M\ell(y,t) + \tau(\xi,t)$ where $\tau(\cdot)$ is a free function. Selecting $\tau(\xi,t) = \xi$ ensures that assumption (A1) is satisfied. Substituting the above expression into (15) yields the equation $\dot z = -\kappa(y,t)z$ which has uniformly asymptotically stable equilibrium at zero, hence assumption (A2) holds. By substituting the expressions of $\varepsilon(\cdot)$ and $\beta(\cdot)$ (with $\tau(\xi,t) = \xi$) in (14), we obtain (12)–(13). ■

Some observations are in order at this point:

- Eq. (12) is a *reduced-order* observer for system (10): in fact it has lower dimension than the system.
- The observer (12) can be easily tuned to achieve the desired convergence rate by acting on the single gain parameter M.

Remark 3. Note that (11), which is necessary to avoid singularities in (12), exactly corresponds to the observability condition (4) derived studying the singularity of the Extended Output Jacobian.

6. Formation control and closed-loop stability

Note that if the state $s = [\eta \;\; \psi \;\; \varphi]^T$ was perfectly known, then system (9) could be exactly input-state feedback linearized and the asymptotic convergence of s towards a desired state s^{des} guaranteed. The presence of an observer inside the control loop obviously makes the convergence analysis more involved. In Proposition 5, we will study the stability of the closed-loop system arising from the combination of the I&I observer and an input-state feedback control law. In this respect, it is convenient to rewrite system (9) in the form:

$$\dot s_r = F(s)u_L + H(s)u_F, \quad (18)$$

$$\dot\varphi = \omega_L - \omega_F, \quad (19)$$

where $s_r = [\eta \;\; \psi]^T$ is the *reduced* state space vector, $u_L = [v_L \;\; \omega_L]^T$, $u_F = [v_F \;\; \omega_F]^T$ and,

$$F(s) = \begin{bmatrix} \eta^2\cos\psi & 0 \\ \eta\sin\psi & -1 \end{bmatrix},$$

$$H(s) = \begin{bmatrix} -\eta^2\cos\gamma & -\eta^2 d\sin\gamma \\ -\eta\sin\gamma & \eta d\cos\gamma \end{bmatrix}.$$

In the following we will implicitly assume that the control input u_F is computed by the leader and transmitted to the follower.

Proposition 5 (*Control and Closed-loop Stability*). *Consider the system (18)–(19) and suppose that $v_L > 0$ and $|\omega_L| \leq \omega_{Lmax}$, $\omega_{Lmax} > 0$. For a given state estimate $\hat s = [\hat\eta \;\; \psi \;\; \varphi]^T$ (with $\hat\eta > 0$) provided by the observer in Proposition 4 with gain M sufficiently large, the feedback control law,*

$$u_F = H^{-1}(\hat s)(p - F(\hat s)u_L), \quad (20)$$

Fig. 2. (a) Trajectory of the leader and the follower; (b) Observation error $\rho - \hat{\rho}$; (c) Control errors $\rho - \rho^{des}$ and $\psi - \psi^{des}$; (d) Control inputs v_F and ω_F; (e) Bearing angle φ.

with $p \triangleq -K(\hat{s}_r - s_r^{des})$, $K = diag\{k_1, k_2\}$, $k_1, k_2 > 0$, $\hat{s}_r = [\hat{\eta}\ \psi]^T$, *guarantees the asymptotic convergence of the control error* $s_r - s_r^{des}$ *to zero and the locally uniformly ultimate boundedness (UUB) of the internal dynamics* φ.

Proof. Substituting (20) in (18) we obtain the dynamics of the controlled system $\dot{s}_r = F(s)u_L + H(s)H^{-1}(\hat{s})(p(\hat{s}_r) - F(\hat{s})u_L)$. Since s_r^{des} is constant, the dynamics of the control error $e_r = s_r - s_r^{des}$ is,

$$\dot{e}_r = \underbrace{\begin{bmatrix} -k_1(\eta/\hat{\eta})^2 & 0 \\ 0 & -k_2(\eta/\hat{\eta}) \end{bmatrix}}_{A(t)} e_r + \underbrace{\begin{bmatrix} -k_1(\hat{\eta} - \eta)(\eta/\hat{\eta})^2 \\ \omega_L(\eta/\hat{\eta} - 1) \end{bmatrix}}_{b(t)}, \quad (21)$$

where $\hat{s}_r = s_r + [\hat{\eta} - \eta, 0]^T$. To prove that the control error asymptotically converges to zero, we should study the stability of a linear time-varying system with perturbation $b(t)$. Let us first study the stability of the equilibrium point $e_r = 0$ of the *non-perturbed* system. Given the candidate Lyapunov function $V = e_r^T e_r$, we have $\dot{V} = e_r^T \dot{e}_r + \dot{e}_r^T e_r = 2e_r^T A(t) e_r \leq 2\lambda_M \|e_r\|^2 = 2\lambda_M V$ where $\lambda_M = \max\{-k_1(\eta/\hat{\eta})^2, -k_2(\eta/\hat{\eta})\}$. Since $\hat{\eta} > 0$, then $\lambda_M < 0$, which implies that $e_r = 0$ is a globally asymptotically stable equilibrium point for the non-perturbed system. To study the stability of the *perturbed* system, let us consider again the Lyapunov function $V = e_r^T e_r$ for which it results:

$$\begin{aligned} \dot{V} &= 2e_r^T A(t) e_r + 2e_r^T b(t) \leq 2\lambda_M \|e_r\|^2 + 2\|e_r\| \|b(t)\| \\ &\leq 2(1-\theta)\lambda_M \|e_r\|^2 + 2\theta\lambda_M \|e_r\|^2 + 2\|e_r\|\delta, \end{aligned} \quad (22)$$

where $0 < \theta < 1$ and $\|b(t)\| \leq \delta$. From the last inequality in (22) we have,

$$\dot{V} \leq 2(1-\theta)\lambda_M \|e_r\|^2 < 0 \quad \text{if} \quad \delta \leq -\theta\lambda_M \|e_r\|, \ \forall e_r.$$

Since $|\omega_L| \leq \omega_{Lmax}$, we can choose $\delta = |\eta/\hat{\eta} - 1| \sqrt{\omega_{Lmax}^2 + k_1^2 \eta^4 \hat{\eta}^{-2}}$ and rewrite $\delta \leq -\theta\lambda_M \|e_r\|$ as:

$$\|e_r\| \geq -(\theta\lambda_M)^{-1} \left|\eta/\hat{\eta} - 1\right| \sqrt{\omega_{Lmax}^2 + k_1^2 \eta^4 \hat{\eta}^{-2}}. \quad (23)$$

We now study under which conditions (23) is verified, that is, $e_r = 0$ is an asymptotically stable equilibrium point for the perturbed system. If $\hat{\eta}$ rapidly converges to η, we note that inequality (23) reduces to $\|e_r\| \geq 0$, that is always true. This implies that $e_r = 0$ is an asymptotically stable equilibrium point for system (21). Note that due to the exponential convergence of the I&I observer's estimation error to zero, there will exist two positive constants D and C such that $\hat{\eta} \geq D e^{-Ct} + \eta$ or equivalently $\left|1 - \hat{\eta}/\eta\right| \geq (D/\eta) e^{-Ct}$. Using this inequality in (23) and observing that parameter C is proportional to the gain M, we see that the asymptotic convergence of the control error to zero can be always guaranteed by choosing M sufficiently large.

It now remains to show that the internal dynamics φ is locally UUB. Exploiting ω_F from (20), we can rewrite Eq. (19) as $\dot{\varphi} = -\frac{v_L}{d}\sin\varphi - \frac{\sin\gamma}{\hat{\eta}^2 d} k_1 e_r(1) + \frac{\cos\gamma}{\hat{\eta} d} k_2 e_r(2) - \omega_L\left(\frac{\cos\gamma}{\hat{\eta} d} - 1\right)$ or more synthetically as $\dot{\varphi} = -\frac{v_L}{d}\sin\varphi + B(t, \varphi)$, where $B(t, \varphi)$ is a nonvanishing perturbation acting on the nominal system $\dot{\varphi} = -\frac{v_L}{d}\sin\varphi$. The nominal system has a locally uniformly asymptotically stable equilibrium point in $\varphi = 0$ and its Lyapunov function $V = \frac{1}{2}\varphi^2$ satisfies the inequalities (Khalil, 2002): $\alpha_1(|\varphi|) \leq V \leq \alpha_2(|\varphi|)$, $-\frac{\partial V}{\partial\varphi}\frac{v_L}{d}\sin\varphi \leq -\alpha_3(|\varphi|)$, $\left|\frac{\partial V}{\partial\varphi}\right| \leq \alpha_4(|\varphi|)$ in $[0, \infty) \times G$, where $G = \{\varphi \in \mathbb{R} : |\varphi| < \epsilon\}$, being ϵ a sufficiently small positive constant. $\alpha_i(\cdot)$, $i = 1, \ldots, 4$, are class $\mathcal{K}$ functions defined as follows: $\alpha_1 = \frac{1}{4}\varphi^2$, $\alpha_2 = \varphi^2$, $\alpha_3 = \frac{v_L}{d}\varphi^2$ and $\alpha_4 = 2|\varphi|$. Since e_r is asymptotically convergent to zero and, by hypothesis ω_L is bounded, there exist suitable velocities for the leader such that $B(t, \varphi)$ satisfies the uniform