

- [30] H. Xiong, "Geometric mechanics, ideal hydrodynamics, and the locomotion of planar shape-changing aquatic vehicles," Ph.D. dissertation, Univ. Illinois at Urbana-Champaign, 2007.
- [31] C. Zhang, D. Arnold, N. Ghods, A. Siranosian, and M. Krstic, "Source seeking with nonholonomic unicycle without position measurement and with tuning of forward velocity," *Syst. Control Lett.*, vol. 56, pp. 245–252, 2007.
- [32] C. Zhang, A. Siranosian, and M. Krstic, "Extremum seeking for moderately unstable systems and for autonomous vehicle target tracking without position meas.," *Automatica*, vol. 43, pp. 1832–1839, 2007.
- [33] X. Zhang, D. Dawson, W. Dixon, and B. Xian, "Extremum seeking nonlinear controllers for a human exercise machine," in *Proc. IEEE Conf. Decis. Control*, 2004, pp. 3950–3955.

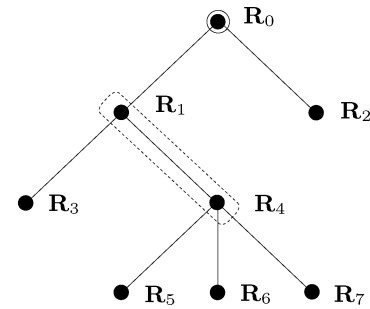


Fig. 1. Structure of a simple hierarchical formation. The *main leader* R_0 guides the formation. One of the seven *leader-follower units* is highlighted: R_1 is the *relative leader* of R_4 . Robots R_5 , R_6 , and R_7 act only as followers in the formation.

Stabilization of a Hierarchical Formation of Unicycle Robots with Velocity and Curvature Constraints

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Abstract—The paper proposes a new geometric approach to the stabilization of a hierarchical formation of unicycle robots. Hierarchical formations consist of elementary leader–follower units disposed on a rooted tree: each follower sees its relative leader as a fixed point in its own reference frame. Robots’ linear velocity and trajectory curvature are forced to satisfy some given bounds. The major contribution of the paper is to study the effect of these bounds on the admissible trajectories of the main leader. In particular, we provide recursive formulas for the maximum velocity and curvature allowed for the main leader, so that the robots can achieve the desired formation while respecting their input constraints. An original formation control law is proposed and the asymptotic stabilization is proved. Simulation experiments illustrate the theory and show the effectiveness of the proposed designs.

Index Terms—Formation control, mobile robots, motion control, multiagent systems, nonlinear systems.

I. INTRODUCTION

Recent years have witnessed a growing interest in robotics, in motion coordination and cooperative control of multiagent systems [1]–[3]. In this respect, several new problems, such as, e.g., *consensus* [4], [5], *rendezvous* [6], [7], *coverage* [8], *connectivity maintenance* [9], [10], and *formation control*, have been formulated and solved using tools coming from computer science and control theory. Among them, for its wide range of applicability, the formation control problem received a special attention and stimulated a great deal of research [11]–[15].

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By formation control, we simply mean the problem of controlling the relative position and orientation of the robots in a group while allowing the group to move as a whole. Typical working scenarios of robot formations are terrain and utilities inspection, disaster monitoring, environmental surveillance, search and rescue, structures moving and assembling, and planetary exploration. Research on formation control dealt with ground vehicles [16], [17], autonomous underwater vehicles (AUVs) [18], [19], unmanned aerial vehicles (UAVs) [20], [21], and microsatellites [22], [23].

One of the main approaches to formation control is *leader-following* [24]–[26]. A robot of the formation, designed as the leader, moves along a predefined trajectory, while the other robots, the followers, are to maintain a desired distance and orientation to the leader. Leader–follower architectures are known to have poor disturbance rejection properties. In addition, the overreliance on a single agent to achieve the goal may be undesirable, especially in adverse conditions. Nevertheless, the leader–follower approach is particularly appreciated for its simplicity and scalability.

This paper extends our previous work [27] on the stabilization of a leader–follower pair of robots, to *hierarchical formations* of unicycles with input constraints. Hierarchical formations are characterized by elementary *leader–follower units*, whose interconnection is described by a rooted tree graph (see Fig. 1). The *main leader* R_0 drives the formation, while all the other robots R_i act both as followers and leaders (with the exception of the leaves of the tree, which are only followers).

The original contribution of this paper with respect to the existing literature is twofold.

1) Each follower in the formation has to maintain a desired distance and orientation to its *relative leader* with respect to its *own local* reference frame.

2) The velocity and the curvature admissible for each follower in the formation are assumed to be *bounded*. The maximum velocity and curvature that are allowed for the main leader are determined through recursive formulas and a stabilizing controller is designed for each robot, so that the desired formation is asymptotically achieved and the input constraints are respected.

A peculiar feature of the proposed control strategy is that at steady state the whole formation is not rigid but changes its shape according to the motion of the main leader. Each follower’s position is not fixed in the reference frame of its relative leader, but it varies in time in a suitable circle arc fixed in the relative leader’s frame.

The rest of the paper is organized as follows. In Section II, some basic definitions are provided and the problem studied in the paper is formulated. In Sections III and IV, an original solution to the stabilization

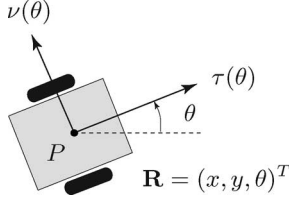


Fig. 2. Variables of the unicycle robot.

problem of a two-robot and a multirobot formation is proposed. In Section V, simulation experiments illustrate the theory and show the effectiveness of the proposed designs. In Section VI, the main contributions of the paper are summarized and future research directions are highlighted.

The following notation will be used throughout the paper:

$\forall a, b \in \mathbb{R}, a \wedge b = \min\{a, b\}, a \vee b = \max\{a, b\}; \forall t > 0, \text{sign}(t) = 1; \forall t < 0, \text{sign}(t) = -1; \forall x = (x_1, \dots, x_n)^T, y = (y_1, \dots, y_n)^T \in \mathbb{R}^n (n \geq 1), \langle x, y \rangle = \sum_{i=1}^n x_i y_i, \|x\| = \sqrt{\langle x, x \rangle}; \forall \theta \in \mathbb{R}, \tau(\theta) = (\cos \theta, \sin \theta)^T, \nu(\theta) = (-\sin \theta, \cos \theta)^T; \mathcal{S}^1$ denotes the quotient space \mathbb{R}/\mathcal{R} equipped with the canonical topology, \mathcal{R} being the equivalence relation $x \sim y \Leftrightarrow x - y = 2m\pi, m \in \mathbb{Z}$.

II. BASIC DEFINITIONS AND PROBLEM FORMULATION

Consider the following definition of robot as a velocity-controlled unicycle model.

Definition 1: $\mathbf{R} = (x, y, \theta)^T \in \mathcal{C}^1([0, +\infty), \mathbb{R}^3)$ is called a *unicycle robot* with initial condition $\bar{\mathbf{R}} \in \mathbb{R}^3$ and control $(v, \omega)^T \in \mathcal{C}^0([0, +\infty), \mathbb{R}^2)$, if the following system is verified:

$$\begin{cases} \dot{x} = v \cos \theta \\ \dot{y} = v \sin \theta \\ \dot{\theta} = \omega \\ (x(0), y(0), \theta(0))^T = \bar{\mathbf{R}}. \end{cases} \quad (1)$$

System (1) represents the differential kinematics of the unicycle robot, and $\forall t \geq 0$ such that $v(t) \neq 0, \kappa(t) = \omega(t)/v(t)$ is the curvature of the path followed by the robot at time t .

Let us denote the position of \mathbf{R} at time t by $P(t) = (x(t), y(t))^T, \theta(t)$ its heading, $\tau(\theta(t))$ the normalized velocity vector, and $\nu(\theta(t))$ the normalized vector orthogonal to $\tau(\theta(t))$. Hence, $\{\tau(\theta(t)), \nu(\theta(t))\}$ represents the robot reference frame at time t (see Fig. 2).

Definition 2: The triple (D, Φ, \mathcal{N}) , where $D = (d_1, d_2, \dots, d_n)^T \in \mathbb{R}^n, \Phi = (\phi_1, \phi_2, \dots, \phi_n)^T \in \mathbb{R}^n$, and $\mathcal{N} = (l_1, l_2, \dots, l_n)^T \in \mathbb{N}^n$, is said to be *admissible* (to make a formation), if $\forall i = 1, \dots, n, d_i > 0, |\phi_i| < \pi/2$, and $l_i \in \{0, \dots, i-1\}$.

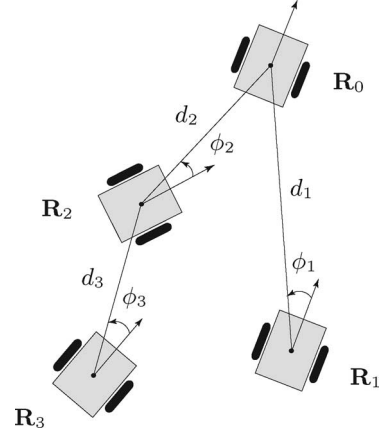
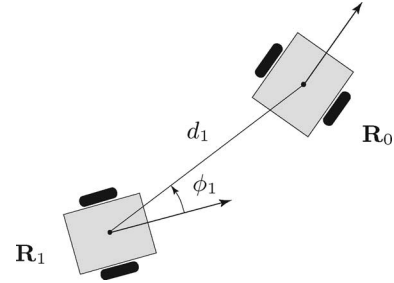
The following definition introduces the notion of *hierarchical formation* (or (D, Φ, \mathcal{N}) -formation) used throughout the paper (see Fig. 3).

Definition 3: Let $(D, \Phi, \mathcal{N}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}^n$ be an admissible triple. We say that $n+1$ robots $\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_n$ are in (D, Φ, \mathcal{N}) -formation with leader \mathbf{R}_0 at time t , if $\forall i = 1, \dots, n$,

$$P_i(t) = P_i(t) + d_i \tau(\theta_i(t) + \phi_i) \quad (2)$$

and simply that $\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_n$ are in (D, Φ, \mathcal{N}) -formation with leader \mathbf{R}_0 , if (2) holds $\forall i = 1, \dots, n, \forall t \geq 0$.

\mathbf{R}_0 guides the formation and is called the *main leader*. \mathbf{R}_{l_i} is the *relative leader* to the i th follower \mathbf{R}_i , and each pair $(\mathbf{R}_{l_i}, \mathbf{R}_i)$ is referred to as a *leader-follower unit*. Definition 3 states that $n+1$ robots $\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_n$ are in (D, Φ, \mathcal{N}) -formation with leader \mathbf{R}_0 , if for any robot \mathbf{R}_i , the position P_i of the follower \mathbf{R}_i is always at distance d_i from the position P_{l_i} of the relative leader \mathbf{R}_{l_i} , and the

Fig. 3. Sample (D, Φ, \mathcal{N}) -formation with $D = (d_1, d_2, d_3)^T, \Phi = (\phi_1, \phi_2, \phi_3)^T$, and $\mathcal{N} = (0, 0, 2)^T$.Fig. 4. $(d_1, \phi_1, 0)$ -formation.

angle between vectors $\tau(\theta_i)$ and $P_i - P_{l_i}$ is constantly equal to ϕ_i , i.e., the position P_{l_i} of the relative leader remains fixed with respect to the reference frame $\{\tau(\theta_i), \nu(\theta_i)\}$. Notice that we use the expression *hierarchical formation* because giving \mathcal{N} is equivalent to assign a *relative leader* to each robot and this defines a partial ordering in the set of the robots, which is usually called a *hierarchy*. Note also that usually in the literature, the follower's position is kept fixed with respect to the frame of the leader (see, e.g., [24] and [26]). On the contrary, in the proposed setup, it is the relative leader's position that is kept fixed in each follower's reference frame.

An equivalent way to state Definition 3 is the following. Set the error vector

$$E_i(t) = P_i(t) - (P_i(t) + d_i \tau(\theta_i(t) + \phi_i))$$

then $\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_n$ are in (D, Φ, \mathcal{N}) -formation with leader \mathbf{R}_0 if and only if $E_i(t) = 0 \forall i = 1, \dots, n, \forall t \geq 0$.

It is worth noting that hierarchical formations represent rooted trees in the context of graph theory. In the special case that $l_i = 0$ for all i , all the robots have \mathbf{R}_0 as relative leader, while if $l_i = i-1$ for all i , a "convoy-like" formation is obtained, i.e., a formation where the i th robot has the $(i-1)$ th as relative leader. Finally, note that if $n = 1$, then $D = d_1, \Phi = \phi_1, \mathcal{N} = 0$, i.e., \mathbf{R}_0 and \mathbf{R}_1 are in $(d_1, \phi_1, 0)$ -formation with leader \mathbf{R}_0 , which is the case considered in [27] (see Fig. 4).

Definition 4: Let $(D, \Phi, \mathcal{N}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}^n$ be an admissible triple. We say that $n+1$ robots $\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_n$ are *asymptotically* in (D, Φ, \mathcal{N}) -formation with leader \mathbf{R}_0 , if $\forall i = 1, \dots, n$,

$$\lim_{t \rightarrow \infty} E_i(t) = P_i(t) - (P_i(t) + d_i \tau(\theta_i(t) + \phi_i)) = 0.$$

The following problem will be studied in the next sections.

Problem 1: Let $(D, \Phi, \mathcal{N}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}^n$ be an admissible triple. Let $\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_n$ be $n+1$ robots and $V_i, K_i^-, K_i^+, i = 0, \dots, n$, be real constants. Find the velocity and curvature bounds V_0, K_0^-, K_0^+ of robot \mathbf{R}_0 that guarantee the existence of controls $v_i, \omega_i, i = 1, \dots, n$, for robot \mathbf{R}_i , such that for any initial condition $\bar{\mathbf{R}}_i$, the robots $\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_n$ are asymptotically in (D, Φ, \mathcal{N}) -formation with leader \mathbf{R}_0 and the following constraints are satisfied: $\forall i = 1, \dots, n$, and $\forall t \geq 0$,

$$0 < v_i(t) \leq V_i, \quad K_i^- \leq \kappa_i(t) \leq K_i^+.$$

Roughly speaking, given the curvature and velocity bounds of each follower robot, we aim to find the bounds on the velocity and curvature of the main leader such that there exist controls for the followers, for which the robots asymptotically achieve the desired formation while respecting their own velocity and curvature constraints.

III. STABILIZATION OF A TWO-ROBOT FORMATION

Before dealing with Problem 1, it is worth studying a simpler case, namely the asymptotic stabilization of a *two-robot formation*. To understand the choice of the controls we are going to use, consider the following remark.

Remark 1: Suppose that two robots \mathbf{R}_0 and \mathbf{R}_1 are in $(d_1, \phi_1, 0)$ -formation with leader \mathbf{R}_0 . Then, it must be

$$v_1(t) = v_0(t) \frac{\cos(\beta_1(t) - \phi_1)}{\cos \phi_1}, \quad \omega_1(t) = v_0(t) \frac{\sin \beta_1(t)}{d_1 \cos \phi_1} \quad (3)$$

where $\beta_1(t) \triangleq \theta_0(t) - \theta_1(t)$ is the relative heading of the two robots. In fact, since $E_1(t) = P_0(t) - (P_1(t) + d_1 \tau(\theta_1(t) + \phi_1)) = 0$, differentiating it, we get that $\dot{P}_0(t) = \dot{P}_1(t) + d_1 \dot{\theta}_1(t) \nu(\theta_1(t) + \phi_1)$, i.e., $v_0(t) \tau(\theta_0(t)) = v_1(t) \tau(\theta_1(t) + \phi_1) + d_1 \omega_1(t) \nu(\theta_1(t) + \phi_1)$, which implies, multiplying by the rotation matrix $\mathcal{R}(-\theta_1(t)) \triangleq (\tau(-\theta_1(t)), \nu(-\theta_1(t)))$, that $v_0(t) \tau(\beta_1(t)) = v_1(t)(1, 0)^T + d_1 \omega_1(t) \nu(\phi_1)$. Therefore, it has to be

$$v_1 = v_0 \left[\cos \beta_1 + \frac{\sin \beta_1 \sin \phi_1}{\cos \phi_1} \right] = v_0 \frac{\cos(\beta_1 - \phi_1)}{\cos \phi_1}$$

$$\omega_1 = v_0 \frac{\sin \beta_1}{d_1 \cos \phi_1}.$$

The stabilization control strategy that we are going to present consists of two steps. In the first step, \mathbf{R}_1 moves with maximum linear and angular velocities until its direction is sufficiently close to that of \mathbf{R}_0 . In the second step, \mathbf{R}_1 performs the control defined in Remark 1 with an added stabilizing term in order to reduce the error asymptotically to zero. Note that since $\forall a, a_1, a_2 \in \mathbb{R}^2$ such that $\langle a_1, a_2^\perp \rangle \neq 0$, $a = (\langle a, a_2^\perp \rangle / \langle a_1, a_2^\perp \rangle) a_1 + (\langle a, a_1^\perp \rangle / \langle a_2, a_1^\perp \rangle) a_2$, (a_1^\perp and a_2^\perp are vectors orthogonal to a_1 and a_2 , respectively), then the error vector E_1 can be decomposed with respect to vectors $\tau(\theta_1)$ and $\nu(\theta_1 + \phi_1)$ as follows:

$$E_1 = \frac{\langle E_1, \nu(\theta_1 + \phi_1)^\perp \rangle}{\langle \tau(\theta_1), \nu(\theta_1 + \phi_1)^\perp \rangle} \tau(\theta_1) + \frac{\langle E_1, \tau(\theta_1)^\perp \rangle}{\langle \nu(\theta_1 + \phi_1), \tau(\theta_1)^\perp \rangle} \nu(\theta_1 + \phi_1)$$

$$= E_{1\tau} \tau(\theta_1) + E_{1\nu} \nu(\theta_1 + \phi_1) \quad (4)$$

where $E_{1\tau} = \langle E_1, \tau(\theta_1 + \phi_1) \rangle / \cos \phi_1$, $E_{1\nu} = \langle E_1, \nu(\theta_1) \rangle / \cos \phi_1$ (see Fig. 5).

Let $\Gamma_\epsilon^1 = \{\gamma \in \mathcal{S}^1 \mid (K_0^- - \epsilon) d_1 \cos \phi_1 \leq \sin \gamma \leq (K_0^+ + \epsilon) d_1 \cos \phi_1\}$. The stabilizing controller that will be referred to in

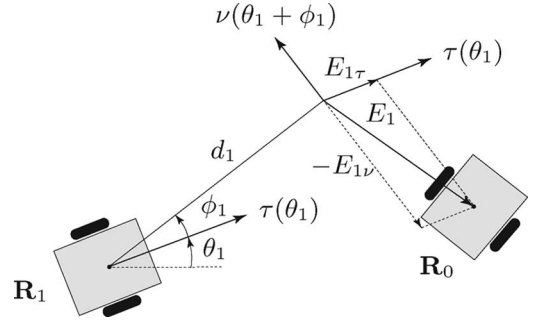


Fig. 5. Error decomposition, where $E_{1\tau} = \langle E_1, \tau(\theta_1 + \phi_1) \rangle / \cos \phi_1$ and $E_{1\nu} = \langle E_1, \nu(\theta_1) \rangle / \cos \phi_1$.

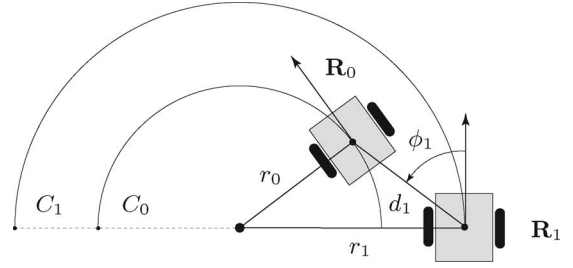


Fig. 6. \mathbf{R}_0 and \mathbf{R}_1 in $(d_1, \phi_1, 0)$ -formation on two circles of radius r_0 and r_1 .

Proposition 1 is given by

$$v_1(t) = \begin{cases} V_1, & \text{if } \beta_1(t) \notin \Gamma_\epsilon^1 \\ \frac{v_0(t) \cos(\beta_1(t) - \phi_1)}{\cos \phi_1} + \eta_1(t) E_{1\tau}(t), & \text{if } \beta_1(t) \in \Gamma_\epsilon^1 \end{cases} \quad (5)$$

$$\omega_1(t) = \begin{cases} V_1 K_1^+, & \text{if } \beta_1(t) \notin \Gamma_\epsilon^1 \text{ and } K_1^+ \geq 0 \\ V_1 K_1^-, & \text{if } \beta_1(t) \notin \Gamma_\epsilon^1 \text{ and } K_1^+ < 0 \\ \frac{v_0(t) \sin \beta_1(t)}{d \cos \phi_1} + \eta_1(t) \frac{E_{1\nu}(t)}{d_1}, & \text{if } \beta_1(t) \in \Gamma_\epsilon^1 \end{cases} \quad (6)$$

which adds correcting terms proportional to the error components to the controls v_1, ω_1 in (3) [see the proof of Proposition 1 in the Appendix for the definition of the proportionality factor $\eta_1(t)$]. Note that by decomposing the error E_1 into the components $E_{1\tau}$ and $E_{1\nu}$, we decoupled the stabilizing control of robot \mathbf{R}_1 . In fact, in (5) v_1 only depends on $E_{1\nu}$ and in (6) ω_1 only depends on $E_{1\tau}$. In the Appendix, we will prove that Γ_ϵ^1 has the following property: if $\beta_1(\bar{t}) \in \Gamma_\epsilon^1$, then, $\beta_1(t) \in \Gamma_\epsilon^1 \forall t \geq \bar{t}$, i.e., Γ_ϵ^1 is a positive controlled invariant set for β_1 . The first step of the stabilization strategy ensures that there exists a time \bar{t} such that $\beta_1(\bar{t}) \in \Gamma_\epsilon^1$. In this way, at time \bar{t} the heading of \mathbf{R}_1 is sufficiently close to the heading of \mathbf{R}_0 . The second step of the control strategy guarantees that β_1 remains in Γ_ϵ^1 and that the two robots asymptotically converge to the desired formation.

To understand the genesis of some of the constants and constraints that appear in the statement of Proposition 1, consider the situation presented in Fig. 6, where the robots \mathbf{R}_0 and \mathbf{R}_1 are in $(d_1, \phi_1, 0)$ -formation and follow two circles C_0 and C_1 of radius r_0, r_1 and curvature $K_0 > 0, K_1 > 0$, respectively. If $d_1 > 0, |\phi_1| < \pi/2$ and

$K_0 < 1/(d_1 \cos \phi_1)$, the following relations hold true:

$$\begin{aligned} r_1 &= d_1 \sin \phi_1 + \sqrt{r_0^2 - d_1^2 \cos^2 \phi_1} \\ K_1 &= \frac{1}{d_1 \sin \phi_1 + \sqrt{(1/K_0^2) - d_1^2 \cos^2 \phi_1}}. \end{aligned} \quad (7)$$

Indeed, as shown in Proposition 1, the value of K_1 given in (7) is precisely the maximum positive curvature of a follower robot whose leader is following a path of maximum positive curvature less than K_0 . An analogous reasoning can be made for negative curvatures. The following definition is introduced to provide concise statements in Proposition 1.

Definition 5: Let $d > 0$ and $\phi : |\phi| < \pi/2$. Define the following strictly increasing monotone functions: $\forall, \kappa \in \mathbf{R}$ such that $|\kappa| \leq 1/(d \cos \phi)$, set,

$$\chi_{d,\phi}(\kappa) = \frac{1}{d \sin \phi + (\text{sign } \kappa) \sqrt{(1/\kappa^2) - d^2 \cos^2 \phi}}$$

and $\forall \kappa \in \mathbf{R}$, set,

$$\chi_{d,\phi}^{-1}(\kappa) = \frac{\text{sign } \kappa}{\sqrt{d^2 \cos^2 \phi + ((1/\kappa) - d \sin \phi)^2}}$$

with the convention that $\chi_{d,\phi}(0) = \chi_{d,\phi}^{-1}(0) = 0$.

Therefore, in the example of Fig. 6, $K_1 = \chi_{d_1,\phi_1}(K_0)$ [see (7)] and $K_0 = \chi_{d_1,\phi_1}^{-1}(K_1)$.

Let $d_1 > 0$, $\phi_1 : |\phi_1| < \pi/2$, and let $W_0, V_0, K_0^-, K_0^+, V_1, K_1^-, K_1^+$ be suitable constants such that

$$\begin{aligned} 0 < W_0, V_0, V_1, \quad K_1^- \leq K_1^+ \\ \left. \begin{aligned} \text{if } \phi_1 < 0, & -\frac{1}{d_1 \cos \phi_1} \\ \text{if } \phi_1 \geq 0, & -\frac{1}{d_1} \end{aligned} \right\} < K_0^- \leq K_0^+ < \left\{ \begin{aligned} \frac{1}{d_1 \cos \phi_1}, & \text{ if } \phi_1 > 0 \\ \frac{1}{d_1}, & \text{ if } \phi_1 \leq 0. \end{aligned} \right. \end{aligned} \quad (8)$$

Set $\check{K}_0^\pm = \chi_{d_1,\phi_1}^{-1}(K_1^\pm)$ and

$$\begin{aligned} \check{V}_0 &= V_1 \cos \phi_1 (\cos(0 \wedge (\arcsin(K_0^+ d_1 \cos \phi_1) - \phi_1)) \\ &\quad \wedge (\phi_1 - \arcsin(K_0^- d_1 \cos \phi_1)))^{-1}. \end{aligned}$$

The proof of the next proposition is given in the Appendix.

Proposition 1: In the previous notation and hypotheses, let \mathbf{R}_0 be a robot that satisfies the following constraints for all $t \geq 0$:

$$0 < W_0 \leq v_0(t) \leq V_0, \quad K_0^- \leq \kappa_0(t) \leq K_0^+. \quad (9)$$

If the following conditions are satisfied:

$$\check{K}_0^- < K_0^- \leq K_0^+ < \check{K}_0^+, \quad V_0 < \check{V}_0 \quad (10)$$

then for any initial condition $\bar{\mathbf{R}}_1$ of the robot \mathbf{R}_1 , the controls v_1, ω_1 given by (5) and (6) are such that

$$\lim_{t \rightarrow \infty} E_1(t) = 0 \quad (11)$$

and the following constraints are verified for robot \mathbf{R}_1 :

$$0 < W_1 \leq v_1(t) \leq V_1, \quad K_1^- \leq \kappa_1(t) \leq K_1^+ \quad \forall t \geq 0 \quad (12)$$

where

$$\begin{aligned} W_1 &= \frac{W_0}{2 \cos \phi_1} \cos((\arcsin((K_0^+ + \epsilon) d_1 \cos \phi_1) - \phi_1) \\ &\quad \vee (\phi_1 - \arcsin((K_0^- - \epsilon) d_1 \cos \phi_1))) \end{aligned} \quad (13)$$

and $\epsilon > 0$ is a sufficiently small constant. Furthermore, $\forall \epsilon > 0, \exists \bar{t} \geq 0 : \forall t \geq \bar{t}$

$$\begin{aligned} \arcsin((K_0^- - \epsilon) d_1 \cos \phi_1) &\leq \theta_0(t) - \theta_1(t) \\ &\leq \arcsin((K_0^+ + \epsilon) d_1 \cos \phi_1). \end{aligned} \quad (14)$$

Proposition 1 states that given a follower robot \mathbf{R}_1 with velocity and curvature constrained by (12), if the leader \mathbf{R}_0 maneuvers with velocity and curvature sufficiently bounded, then the control law given by (5) and (6) allows \mathbf{R}_1 to asymptotically reach the formation for every initial state. The bounds on the leader's velocity and curvature [see conditions (8) and (10)] depend on the formation's parameters d_1, ϕ_1 , and on the follower's capability of maneuver.

Note that condition (14) states that the difference $\theta_0(t) - \theta_1(t)$ between the robots' headings remains bounded during the motion. The following remark gives a geometric interpretation to this property.

Remark 2: Let

$$\mathcal{A}_d(\delta, \sigma_1, \sigma_2) = \{d\tau(\theta) \mid \delta + \sigma_1 \leq \theta \leq \delta + \sigma_2\}$$

be the arc of circle centered in the origin, radius d , angle of the reference axis δ , aperture $\sigma_2 - \sigma_1$, and let $\mathcal{A}_d^\epsilon(\delta, \sigma_1, \sigma_2) = \{z \in \mathbb{R}^2 \mid \exists z' \in \mathcal{A}_d(\delta, \sigma_1, \sigma_2) : \|z' - z\| < \epsilon\}$ be the ϵ -neighborhood of $\mathcal{A}_d(\delta, \sigma_1, \sigma_2)$. Since, $\forall t \geq 0$,

$$\begin{aligned} P_1(t) &= P_0(t) - d_1 \tau(\theta_1(t) + \phi_1) - E_1(t) \\ &= P_0(t) + d_1 \tau(\theta_1(t) + \phi_1 + \pi) - E_1(t) \\ &= P_0(t) + d_1 \tau((\theta_0(t) + \phi_1 + \pi) - (\theta_1(t) - \theta_0(t))) - E_1(t) \end{aligned}$$

we immediately deduce from (11) and (14) that the following property holds true:

$$\begin{aligned} \forall \epsilon > 0, \exists \bar{t} : \forall t \geq \bar{t}, \quad P_1(t) \in P_0(t) + \mathcal{A}_{d_1}^\epsilon(\theta_0(t) + \phi_1 + \pi, \\ - \arcsin(K_0^+ d_1 \cos \phi_1), - \arcsin(K_0^- d_1 \cos \phi_1)). \end{aligned}$$

In other words, even if P_1 is not, in general, fixed in the leader's reference frame $\{\tau(\theta_0(t)), \nu(\theta_0(t))\}$ during its motion, given any $\epsilon > 0$, $P_1(t)$ belongs eventually to the ϵ -neighborhood of the arc of circle $\mathcal{A}_{d_1}(\theta_0(t) + \phi_1 + \pi, - \arcsin(K_0^+ d_1 \cos \phi_1), - \arcsin(K_0^- d_1 \cos \phi_1))$, which is fixed with respect to the reference frame of robot \mathbf{R}_0 (see Fig. 7, where $\sigma_1 = - \arcsin(K_0^+ d_1 \cos \phi_1)$, $\sigma_2 = - \arcsin(K_0^- d_1 \cos \phi_1)$, and $\delta = \theta_0(t) + \phi_1 + \pi$ in the case of $K_0^- < 0 < K_0^+$).

Remark 3: In [27, Th. 1], it has been proved that conditions (8) and (10) with the weak inequalities are necessary and sufficient for the existence of controls that satisfy the constraints and maintain the vehicles exactly in $(d_1, \phi_1, 0)$ -formation, i.e., $E_1(t) = 0 \forall t \geq 0$.

IV. STABILIZATION OF A MULTIROBOT FORMATION

In this section, we extend the results presented in Section III to multi-robot formations. The following definitions are introduced to provide concise statements in Theorem 1.

Definition 6: Let $(D, \Phi, \mathcal{N}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}^n$ be an admissible triple. For every $i = 0, \dots, n$, let

$$\mathcal{L}_i = \{j \mid i < j \leq n \text{ and } l_j = i\}$$

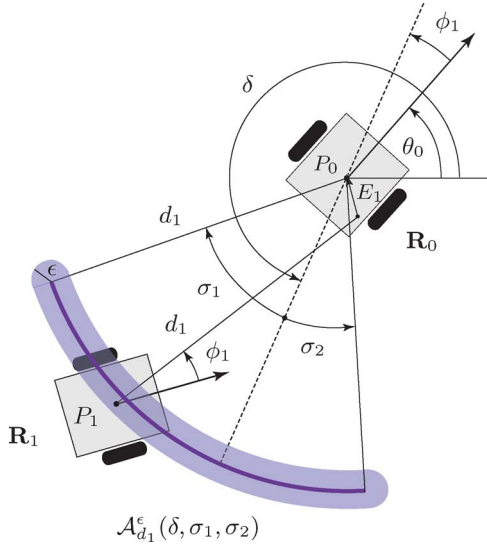


Fig. 7. ϵ -Neighborhood of the arc of circle $\mathcal{A}_{d_1}(\delta, \sigma_1, \sigma_2)$.

be the set of the indexes j such that \mathbf{R}_i is the relative leader of \mathbf{R}_j . Recall that $\forall i = 1, \dots, n$, l_i denotes the index of the robot that is the leader of \mathbf{R}_i .

Note that \mathbf{R}_i is a relative leader if and only if $\mathcal{L}_i \neq \emptyset$, and it always holds that $\mathcal{L}_0 \neq \emptyset$ and $\mathcal{L}_n = \emptyset$.

Definition 7: Let $K_i^-, K_i^+, i = 1, \dots, n$, be $2n$ real constants. Set $\forall i = n, n-1, \dots, 1$,

$$\begin{aligned} \check{K}_i^- &= \max\{K_i^-, \chi_{d_j, \phi_j}^{-1}(\check{K}_j^-) \mid j \in \mathcal{L}_i\} \\ \check{K}_i^+ &= \min\{K_i^+, \chi_{d_j, \phi_j}^{-1}(\check{K}_j^+) \mid j \in \mathcal{L}_i\} \end{aligned}$$

and

$$\begin{aligned} \check{K}_0^- &= \max\{\chi_{d_j, \phi_j}^{-1}(\check{K}_j^-) \mid j \in \mathcal{L}_0\} \\ \check{K}_0^+ &= \min\{\chi_{d_j, \phi_j}^{-1}(\check{K}_j^+) \mid j \in \mathcal{L}_0\}. \end{aligned}$$

Note that $\check{K}_i^\pm = K_i^\pm$ if $\mathcal{L}_i = \emptyset$, and therefore, $\check{K}_n^\pm = K_n^\pm$. Moreover, $\check{K}_i^- \leq \check{K}_i^+$ if $K_i^- \leq K_i^+$, by the monotonicity of χ^{-1} .

Definition 8: Let $(D, \Phi, \mathcal{N}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}^n$ be an admissible triple. Suppose that $\forall i = 1, \dots, n-1$,

$$\left. \begin{aligned} &\text{if } \phi_i < 0, \quad -\frac{1}{d_i \cos \phi_i} \\ &\text{if } \phi_i \geq 0, \quad -\frac{1}{d_i} \end{aligned} \right\} < K_{l_i}^- \leq K_{l_i}^+ < \begin{cases} \frac{1}{d_i \cos \phi_i}, & \text{if } \phi_i > 0 \\ \frac{1}{d_i}, & \text{if } \phi_i \leq 0 \end{cases} \quad (15)$$

is satisfied. Then, $\forall i = 0, \dots, n$, we have

$$-\frac{\pi}{2} < \arcsin(K_{l_i}^- d_i \cos \phi_i) - \phi_i < \arcsin(K_{l_i}^+ d_i \cos \phi_i) - \phi_i < \frac{\pi}{2}.$$

Therefore, the following constants are well defined $\forall i = n, n-1, \dots, 1$:

$$\begin{aligned} \check{V}_i &= \min\{V_i, \check{V}_j \cos \phi_j (\cos(0 \wedge (\arcsin(K_{l_i}^+ d_j \cos \phi_j) - \phi_j) \\ &\quad \wedge (\phi_j - \arcsin(K_{l_i}^- d_j \cos \phi_j))))^{-1} \mid j \in \mathcal{L}_i\} \end{aligned}$$

and

$$\begin{aligned} \check{V}_0 &= \min\{\check{V}_j \cos \phi_j (\cos(0 \wedge (\arcsin(K_0^+ d_j \cos \phi_j) - \phi_j) \\ &\quad \wedge (\phi_j - \arcsin(K_0^- d_j \cos \phi_j))))^{-1} \mid j \in \mathcal{L}_0\}. \end{aligned}$$

Note that $\check{V}_i = V_i$ if $\mathcal{L}_i = \emptyset$; therefore, $\check{V}_n = V_n$.

In the following, we will refer to the expressions in Definitions 7 and 8 as *constraints propagation formulas*.

We are now ready to deal with Problem 1 and state the main theorem of the paper. The proof is reported in the Appendix. Given an admissible triple (D, Φ, \mathcal{N}) and $n+1$ robots $\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_n$ with preassigned velocity and curvature constraints, the following theorem presents a method to determine the corresponding bounds on the velocity and curvature of the main leader \mathbf{R}_0 , such that the robots asymptotically achieve the desired formation while respecting their constraints at all times.

Theorem 1 (Main theorem): Let $(D, \Phi, \mathcal{N}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}^n$ be an admissible triple, $\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_n$ be $n+1$ robots, and $V_i > 0$, $K_i^- \leq K_i^+$, $i = 0, \dots, n$, be $3(n+1)$ real constants, such that $\forall i = 1, \dots, n$, condition (15) is satisfied. Suppose that there exists a constant W_0 such that

$$0 < W_0 \leq v_0(t) \leq V_0, \quad K_0^- \leq \kappa_0(t) \leq K_0^+. \quad (16)$$

Let $\check{K}_0^-, \check{K}_0^+$, and \check{V}_0 be as given in Definitions 7 and 8. If the following conditions are satisfied:

$$\check{K}_0^- < K_0^- \leq K_0^+ < \check{K}_0^+, \quad V_0 < \check{V}_0 \quad (17)$$

then there exists $2n$ suitable controls $v_i, \omega_i, i = 1, \dots, n$, such that the robots $\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_n$ are asymptotically in (D, Φ, \mathcal{N}) -formation with leader \mathbf{R}_0 , and the following conditions are fulfilled:

$$0 < W_i \leq v_i(t) \leq V_i, \quad K_i^- \leq \kappa_i(t) \leq K_i^+ \quad \forall t \geq 0 \quad (18)$$

where W_i is given in (29) (see the Appendix). Moreover, $\forall \epsilon > 0, \exists \bar{t} : \forall t \geq \bar{t}$,

$$\begin{aligned} P_i(t) &\in P_{l_i}(t) + \mathcal{A}_{d_i}^\epsilon(\theta_{l_i}(t) + \phi_i + \pi, -\arcsin(K_{l_i}^+ d_i \cos \phi_i) \\ &\quad - \arcsin(K_{l_i}^- d_i \cos \phi_i)) \quad \forall i = 1, \dots, n. \end{aligned} \quad (19)$$

Remark 4: Hypothesis (17) guarantees that the leader's allowed maximum and minimum curvatures K_0^+, K_0^- are strictly contained in the set $[\check{K}_0^-, \check{K}_0^+]$ of leader's curvature values that are compatible with all the followers' bounds and, similarly, that the leader's maximum velocity V_0 is lower than the maximum velocity \check{V}_0 compatible with all the followers' constraints.

Remark 5: As shown in the proof in the Appendix, under the hypotheses of Theorem 1, if we set $\Gamma_\epsilon^i = \{\gamma \in \mathcal{S}^1 \mid (K_{l_i}^- - \epsilon) d_i \cos \phi_i \leq \sin \gamma \leq (K_{l_i}^+ + \epsilon) d_i \cos \phi_i\}$ and $\beta_i(t) = \theta_{l_i}(t) - \theta_i(t)$, the controls

$$v_i(t) = \begin{cases} V_i, & \text{if } \beta_i(t) \notin \Gamma_\epsilon^i \\ \frac{v_{l_i}(t) \cos(\beta_i(t) - \phi_i)}{\cos \phi_i} + \eta_i(t) E_{i\tau}(t), & \text{if } \beta_i(t) \in \Gamma_\epsilon^i \end{cases}$$

$$\omega_i(t) = \begin{cases} V_i K_{l_i}^+, & \text{if } \beta_i(t) \notin \Gamma_\epsilon^i \text{ and } K_{l_i}^+ \geq 0 \\ V_i K_{l_i}^-, & \text{if } \beta_i(t) \notin \Gamma_\epsilon^i \text{ and } K_{l_i}^+ < 0 \\ \frac{v_{l_i}(t) \sin \beta_i(t)}{d_i \cos \phi_i} + \eta_i(t) \frac{E_{i\nu}(t)}{d_i}, & \text{if } \beta_i(t) \in \Gamma_\epsilon^i \end{cases}$$

(where $E_{i\tau}(t) = \langle E_i(t), \tau(\theta_i(t) + \phi_i) \rangle / \cos \phi_i$, $E_{i\nu}(t) = \langle E_i(t), \nu(\theta_i(t)) \rangle / \cos \phi_i$) assure that $\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_n$ are asymptotically in (D, Φ, \mathcal{N}) -formation with leader \mathbf{R}_0 . $\eta_i(t)$ is a bounded continuous function given by

$$\begin{aligned} \eta_i(t) &= \frac{(v_{l_i}(t) - W_{l_i}/2) \cos(\beta_i(t) - \phi_i)}{|\langle E_i(t), \tau(\theta_i(t) + \phi_i) \rangle|} \\ &\quad \wedge \frac{((K_{l_i}^+ + \epsilon/2 - \kappa_{l_i}(t)) \wedge (\kappa_{l_i}(t) - (K_{l_i}^- - \epsilon/2))) d_i \cos \phi_i}{|\langle E_i(t), \nu(\theta_i(t)) \rangle|} \end{aligned}$$

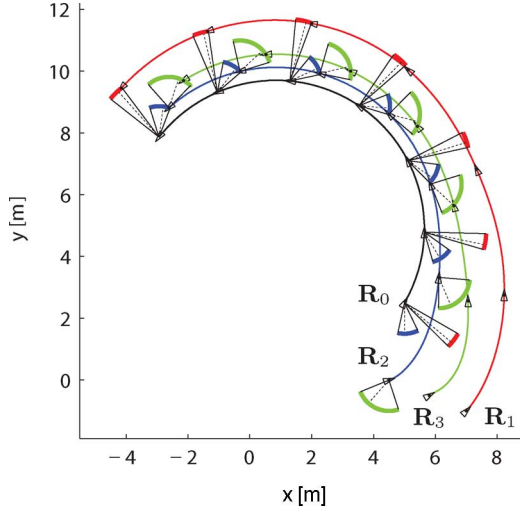


Fig. 8. Trajectory of the robots and arcs of circle $\mathcal{A}_{d_i}^\epsilon(\theta_i(t) + \phi_i + \pi, -\arcsin(K_i^+ d_i \cos \phi_i), -\arcsin(K_i^- d_i \cos \phi_i))$, $i = 1, 2, 3$.

$$\begin{aligned} & \wedge \frac{V_i \cos \phi_i - v_i(t) \cos(\beta_i(t) - \phi_i)}{|\langle E_i(t), \tau(\theta_i(t) + \phi_i) \rangle|} \\ & \wedge \frac{v_i(t)(K_i^+ d_i \cos(\beta_i(t) - \phi_i) - \sin \beta_i(t))}{|\langle E_i(t), \nu(\theta_i(t)) \rangle| + |K_i^+| |\langle E_i(t), \tau(\theta_i(t) + \phi_i) \rangle|} \\ & \wedge \frac{v_i(t)(\sin \beta_i(t) - K_i^- d_i \cos(\beta_i(t) - \phi_i))}{|\langle E_i(t), \nu(\theta_i(t)) \rangle| + |K_i^-| |\langle E_i(t), \tau(\theta_i(t) + \phi_i) \rangle|} \wedge M_i \end{aligned} \quad (20)$$

where M_i are positive gain constants.

V. SIMULATION EXPERIMENTS

Simulation experiments have been conducted to illustrate the theory and show the effectiveness of the formation controller presented in Section IV.

We considered the (D, Φ, \mathcal{N}) -formation reported in Fig. 3, with $d_1 = 2$ m, $d_2 = d_3 = 1$ m, $\phi_1 = 4\pi/9$ rad and $\phi_2 = \phi_3 = \pi/6$ rad. The initial conditions of the robots are $\mathbf{R}_0 = (5, 2.5, \pi/3)^T$, $\mathbf{R}_1 = (7, -1, \pi/4)^T$, $\mathbf{R}_2 = (4.5, 0, \pi/10)^T$, and $\mathbf{R}_3 = (5.8, -0.5, \pi/6)^T$. The main leader \mathbf{R}_0 moves along a circular path, with velocities $v_0(t) = 1.2$ m/s and $\omega_0(t) = 0.25$ rad/s. We chose $W_0 = 1$ m/s, $V_0 = 1.5$ m/s, $V_1 = V_2 = V_3 = 3$ m/s, $K_0^- = -0.25$ rad/m, $K_0^+ = 0.5$ rad/m, $K_1^- = K_2^- = K_3^- = -0.5$ rad/m, and $K_1^+ = K_2^+ = K_3^+ = 1$ rad/m. With these values, condition (15) is satisfied for $i = 1, 2, 3$, and inequalities (16) hold true. Note that in this case, $\mathcal{L}_0 = \{1, 2\}$, $\mathcal{L}_1 = \mathcal{L}_3 = \emptyset$, and $\mathcal{L}_2 = \{3\}$. Using the constraints propagation formulas given in Definitions 7 and 8, we get the bounds \check{K}_0^- , \check{K}_0^+ , and \check{V}_0 . It is easy to verify that \check{K}_0^- , \check{K}_0^+ , and \check{V}_0 satisfy condition (17). The gains M_1 , M_2 , and M_3 in (20) were set to 1. Fig. 8 shows the trajectory of the robots and the arcs of circle $\mathcal{A}_{d_i}^\epsilon(\theta_i(t) + \phi_i + \pi, -\arcsin(K_i^+ d_i \cos \phi_i), -\arcsin(K_i^- d_i \cos \phi_i))$, $i = 1, 2, 3$ are defined in Remark 2 ($\epsilon = 10^{-3}$). In order to have a temporal reference in the figure, the vehicles are drawn in every 2 s. Note that after about 9 s, the robots achieve the desired formation. Fig. 9 shows that the norm of the errors E_1 , E_2 , and E_3 asymptotically converges to zero.

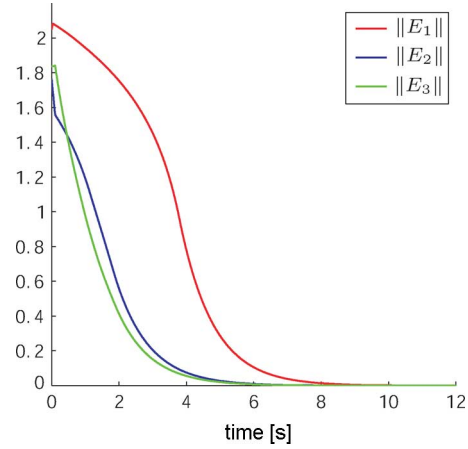


Fig. 9. Norm of the errors E_1 , E_2 , and E_3 .

VI. CONCLUSION AND FUTURE WORK

The paper presents a new geometric approach to the stabilization of a hierarchical formation of unicycle robots. The major contribution of the work is the study of the effect of robots' input constraints on the admissible trajectories of the leader guiding the formation. In particular, we provide recursive formulas for the maximum velocity and curvature of the main leader, such that all the other robots can follow the relative leaders while respecting their input constraints. An original formation control strategy is proposed and the asymptotic stabilization is proved.

Future research lines include the extension of our results to vehicles with more involved kinematics (e.g., car-like robots) and the insertion of communication constraints among the agents into our model.

APPENDIX

A. Proof of Proposition 1

Let us take any $\epsilon > 0$ sufficiently small such that

$$\begin{aligned} & \left. \begin{array}{l} \text{if } \phi_1 < 0, \quad -\frac{1}{d_1 \cos \phi_1} \\ \text{if } \phi_1 \geq 0, \quad -\frac{1}{d_1} \end{array} \right\} < K_0^- - \epsilon \\ & \leq K_0^+ + \epsilon < \begin{cases} \frac{1}{d_1 \cos \phi_1}, & \text{if } \phi_1 > 0 \\ \frac{1}{d_1}, & \text{if } \phi_1 \leq 0 \end{cases} \\ & \check{K}_0^- < K_0^- - \epsilon \leq K_0^+ + \epsilon < \check{K}_0^+ \\ & V_0 \cos(0 \wedge (\arcsin((K_0^+ + \epsilon) d_1 \cos \phi_1) - \phi_1)) \\ & \wedge (\phi_1 - \arcsin((K_0^- - \epsilon) d_1 \cos \phi_1)) < V_1 \cos \phi_1. \end{aligned} \quad (21)$$

Let the controls be given by (5) and (6), where

$$\begin{aligned} \eta_1(t) &= \frac{(v_0(t) - W_0/2) \cos(\beta_1(t) - \phi_1)}{|\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle|} \\ & \wedge \frac{((K_0^+ + \epsilon/2 - \kappa_0(t)) \wedge (\kappa_0(t) - (K_0^- - \epsilon/2))) d_1 \cos \phi_1}{|\langle E_1(t), \nu(\theta_1(t)) \rangle|} \\ & \wedge \frac{V_1 \cos \phi_1 - v_0(t) \cos(\beta_1(t) - \phi_1)}{|\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle|} \\ & \wedge \frac{v_0(t)(K_1^+ d_1 \cos(\beta_1(t) - \phi_1) - \sin \beta_1(t))}{|\langle E_1(t), \nu(\theta_1(t)) \rangle| + |K_1^+| |\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle|} \end{aligned}$$

$$\wedge \frac{v_0(t)(\sin \beta_1(t) - K_1^- d_1 \cos(\beta_1(t) - \phi_1))}{|\langle E_1(t), \nu(\theta_1(t)) \rangle| + |K_1^-| |\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle|} \wedge M_1 \quad (22)$$

with the convention that $1/0 = +\infty$ and M_1 is a positive gain constant. Note that the complexity of $\eta_1(t)$ is due to the necessity of satisfying both the curvature and velocity constraints. First of all, we observe that

$$\exists \bar{t} \geq 0 : \beta_1(\bar{t}) \in \Gamma_\epsilon^1. \quad (23)$$

In fact, suppose, for instance that $K_1^+ \geq 0$; then, if $\forall t \geq 0$, $\beta_1(t) \notin \Gamma_\epsilon^1$, by (10), $\dot{\beta}_1(t) = \omega_0(t) - \omega_1(t) = \kappa_0(t) v_0(t) - K_1^+ V_1 \leq K_0^+ v_0(t) - K_1^+ V_1 \leq (K_0^+ v_0(t) - K_1^+ V_1) - \epsilon v_0(t) \leq (K_0^+ V_0 - K_1^+ V_1) - \epsilon W_0 \leq -\epsilon W_0$, being $(K_0^+ V_0 - K_1^+ V_1) \leq 0$. In fact, suppose for the sake of simplicity that $\cos(0 \wedge (\arcsin((K_0^+ + \epsilon) d_1 \cos \phi_1) - \phi_1) \wedge (\phi_1 - \arcsin((K_0^- - \epsilon) d_1 \cos \phi_1))) = \cos 0 = 1$, by (12) and (21), we have

$$\begin{aligned} \check{K}_0^+ V_0 - K_1^+ V_1 &\leq \frac{V_1 \cos \phi_1}{\sqrt{d_1^2 \cos^2 \phi_1 + ((K_1^+)^{-1} - d_1 \sin \phi_1)^2}} - K_1^+ V_1 \\ &\leq \frac{V_1 (\cos \phi_1 - \sqrt{\cos^2 \phi_1 + (K_1^+ d_1 - \sin \phi_1)^2})}{\sqrt{d_1^2 \cos^2 \phi_1 + ((K_1^+)^{-1} - d_1 \sin \phi_1)^2}} \leq 0. \end{aligned}$$

Therefore, $\beta_1(t) \leq -\epsilon W_0 t + \beta_1(t_0) \quad \forall t \geq 0$, which implies straightaway property (23). Set $\beta_1^{-1}(\Gamma_\epsilon^1) = \{t \geq 0 \mid \beta_1(t) \in \Gamma_\epsilon^1\}$; then, by (6), $\forall t \in \beta_1^{-1}(\Gamma_\epsilon^1)$, we have

$$\begin{aligned} \dot{\beta}_1(t) &= \omega_0(t) - \omega_1(t) \\ &= \omega_0(t) - \frac{v_0(t) \sin \beta_1(t)}{d_1 \cos \phi_1} + \eta_1(t) \frac{E_{1\nu}(t)}{d_1} \\ &= \frac{v_0(t)(\kappa_0(t) d_1 \cos \phi_1 - \sin \beta_1(t))}{d_1 \cos \phi_1} - \eta_1(t) \frac{E_{1\nu}(t)}{d_1} \end{aligned}$$

which implies by definition (22) of $\eta_1(t)$ that if $\beta_1(\bar{t}) = \arcsin((K_0^+ + \epsilon) d_1 \cos \phi_1)$, $(\beta_1(\bar{t}) = \arcsin((K_0^- - \epsilon) d_1 \cos \phi_1))$, then $\dot{\beta}_1(\bar{t}) \leq -\epsilon V_0/2$, $(\dot{\beta}_1(\bar{t}) \geq \epsilon W_0/2)$. This implies, together with (23), that

$$\exists \bar{t} \geq 0 : \forall t \in [0, \bar{t}], \quad \beta_1(t) \notin \Gamma_\epsilon^1 \quad \text{and} \quad \forall t \geq \bar{t}, \quad \beta_1(t) \in \Gamma_\epsilon^1 \quad (24)$$

which gives (14). To prove that constraints (12) are verified for v_1 , note that by (5) and (24), $v_1(t) = V_1 \quad \forall t \in [0, \bar{t}]$ and $\forall t \geq \bar{t}$,

$$v_1(t) = \frac{v_0(t) \cos(\beta_1(t) - \phi_1)}{\cos \phi_1} + \eta_1(t) E_{1\tau}.$$

Therefore, by definition of $\eta_1(t)$ and (9), we have

$$\begin{aligned} v_1(t) &\geq \frac{v_0(t) \cos(\beta_1(t) - \phi_1) - \eta_1(t) |\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle|}{\cos \phi_1} \\ &\geq \frac{W_0 \cos(\beta_1(t) - \phi_1)}{2 \cos \phi_1}. \end{aligned}$$

Hence $v_1(t) > 0 \quad \forall t \geq 0$, by the following property:

$$\exists c_1 > 0 : \cos(\beta_1(t) - \phi_1) \geq c_1 \quad \forall t \geq \bar{t}. \quad (25)$$

In fact, suppose, for instance, that $\phi_1 > 0$, by (14) and (8), we have $\beta_1(t) - \phi_1 \leq \arcsin((K_0^+ + \epsilon) d_1 \cos \phi_1) - \phi_1 < \arcsin(1) - \phi_1 = (\pi/2) - \phi_1$ and $\beta_1(t) - \phi_1 \geq \arcsin((K_0^- - \epsilon) d_1 \cos \phi_1) - \phi_1 > -\arcsin(\cos \phi_1) - \phi_1 = -\pi/2$, which implies (25) and (13). Furthermore, by (5) and (22), we have $v_1(t) \leq V_1 \quad \forall t \geq 0$. To verify constraints (12) for $\kappa_1(t)$, note first of all that they are verified by (5) and (6) if $t \in [0, \bar{t}]$. If $t \geq \bar{t}$, since $\eta_1(t) > 0$ [see (26)], we have $k_1(t)$ as shown at the bottom of this page, because, by (22)

$$\eta_1(t) \leq \frac{v_0(t)(K_1^+ d_1 \cos(\beta_1(t) - \phi_1) - \sin \beta_1(t))}{|\langle E_1(t), \nu(\theta_1(t)) \rangle| + |K_1^+| |\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle|}$$

and analogously, $\kappa_1(t) \geq K_1^- \quad \forall t \geq \bar{t}$; therefore, constraints (12) are completely satisfied. To conclude the proof, it remains to verify (11). Differentiating the error $E_1(t)$, by (5) and (6), and recalling (4), we get that $\forall t \geq \bar{t}$

$$\begin{aligned} \dot{E}_1(t) &= v_0(t) \tau(\theta_0(t)) - v_0(t) \left(\frac{\cos(\beta_1(t) - \phi_1)}{\cos \phi_1} \tau(\theta_1(t)) \right. \\ &\quad \left. + d_1 \frac{\sin \beta_1(t)}{\cos \phi_1} \nu(\theta_1(t) + \phi_1) \right) - \eta_1(t) (E_{1\tau} \tau(\theta_1(t)) \\ &\quad \left. + E_{1\nu} \nu(\theta_1(t) + \phi_1)) = -\eta_1(t) E_1(t). \end{aligned}$$

Therefore, $(\frac{d}{dt}) \|E_1(t)\|^2 = -2\eta_1(t) \|E_1(t)\|^2 \quad \forall t \geq \bar{t}$, and then (11) is satisfied, since the following property holds true:

$$\exists c > 0 : \eta_1(t) \geq \left(\frac{c}{\|E_1(t)\|} \wedge M_1 \right) \quad \forall t \geq \bar{t}. \quad (26)$$

In fact, to verify (26), first of all note that by (9) it follows straightaway that $(K_0^+ + \epsilon/2 - \kappa_0(t)) \wedge (\kappa_0(t) - (K_0^- - \epsilon/2)) \geq \epsilon/2 \quad \forall t \geq 0$. Moreover, by (21), $V_1 \cos \phi_1 - v_0(t) \cos(\beta_1(t) - \phi_1) \geq V_1 \cos \phi_1 - V_0 \cos(0 \wedge (\arcsin(K_0^+ d_1 \cos \phi_1) - \phi_1) \wedge (\phi_1 - \arcsin(K_0^- d_1 \cos \phi_1))) = c_2 > 0$, and furthermore, by (14), $\forall t \geq \bar{t}$, we have

$$\begin{aligned} K_1^+ - \frac{\sin \beta_1(t)}{d_1 \cos(\beta_1(t) - \phi_1)} \\ \geq K_1^+ - \frac{\text{sign}(K_0^+ + \epsilon)}{\sqrt{(1/(K_0^+ + \epsilon)^2) - d_1^2 \cos^2 \phi_1 + d_1 \sin \phi_1}} > 0 \end{aligned}$$

since $(K_0^+ + \epsilon) < \check{K}_0^+$, which implies that $\exists c_3^+ > 0$, such that $K_1^+ d_1 \cos(\beta_1(t) - \phi_1) - \sin \beta_1(t) \geq c_3^+$; analogously, $\exists c_3^- > 0$, such that $\sin \beta_1(t) - K_1^- d_1 \cos(\beta_1(t) - \phi_1) \geq c_3^-$. Therefore, bringing together (25) with the previous inequalities, we obtain (26) by definition of $\eta_1(t)$. ■

B. Proof of Theorem 1

The proof relies on the iterative application of Proposition 1. Let us define the following constants:

$$\begin{aligned} \widehat{K}_0^- &= K_0^-, & \widehat{K}_0^+ &= K_0^+ \\ \widehat{K}_i^- &= \chi_{d_i, \phi_i}(\widehat{K}_i^-), & \widehat{K}_i^+ &= \chi_{d_i, \phi_i}(\widehat{K}_i^+). \end{aligned}$$

$$\begin{aligned} \kappa_1(t) &= \frac{\omega_1(t)}{v_1(t)} = \frac{v_0(t) \sin \beta_1(t) + \eta_1(t) \langle E_1(t), \nu(\theta_1(t)) \rangle}{d_1 v_0(t) \cos(\beta_1(t) - \phi_1) + \eta_1(t) \langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle} \\ &\leq \frac{v_0(t) \sin \beta_1(t) + \eta_1(t) |\langle E_1(t), \nu(\theta_1(t)) \rangle|}{d_1 v_0(t) \cos(\beta_1(t) - \phi_1) - \eta_1(t) (\text{sign } K_1^+) |\langle E_1(t), \tau(\theta_1(t) + \phi_1) \rangle|} \leq K_1^+ \end{aligned}$$

We want to verify that $\forall i = 1, \dots, n$, we have

$$\check{K}_i^- < \widehat{K}_i^-, \quad \widehat{K}_i^+ < \check{K}_i^+. \quad (27)$$

Note that the interval $[\widehat{K}_i^-, \widehat{K}_i^+]$ represents the curvature range necessary for the i th robot to follow its relative leader. On the other hand, the interval $[\check{K}_i^-, \check{K}_i^+]$ represents the curvature range that is compatible with the curvature constraints of the robots contained in the subtree having the robot i th as root. We will prove only the second inequality in (27), since the first one follows analogously. Set $i = 1$, by hypothesis (17) and monotonicity of χ_{d_1, ϕ_1} , we have

$$\begin{aligned} \widehat{K}_1^+ &= \chi_{d_1, \phi_1}(\widehat{K}_0^+) = \chi_{d_1, \phi_1}(K_0^+) \\ &< \chi_{d_1, \phi_1}(\check{K}_0^-) \leq \chi_{d_1, \phi_1}(\chi_{d_1, \phi_1}^{-1}(\check{K}_1^+)) = \check{K}_1^+. \end{aligned}$$

By induction, let suppose that the second in (27) holds for $1 \leq i \leq m-1$ (with $1 < m \leq n$). Since the property holds for $i = l_m$, being $1 \leq l_m \leq m-1$, we have, as before,

$$\begin{aligned} \widehat{K}_m^+ &= \chi_{d_m, \phi_m}(\widehat{K}_{l_m}^+) < \chi_{d_m, \phi_m}(\check{K}_{l_m}^-) \\ &\leq \chi_{d_m, \phi_m}(\chi_{d_m, \phi_m}^{-1}(\check{K}_m^+)) = \check{K}_m^+. \end{aligned}$$

Therefore, the second in (27) holds for $i = m$. For any $\rho > 0$, let us define

$$\widehat{K}_{0, \rho}^- = K_0^-, \quad \widehat{K}_{0, \rho}^+ = K_0^+$$

and $\forall i = 1, \dots, n$

$$\widehat{K}_{i, \rho}^- = \chi_{d_i, \phi_i}(\widehat{K}_{i, \rho}^- - \rho), \quad \widehat{K}_{i, \rho}^+ = \chi_{d_i, \phi_i}(\widehat{K}_{i, \rho}^+ + \rho).$$

By the continuity of χ_{d_i, ϕ_i} , we get that $\lim_{\rho \rightarrow 0} \widehat{K}_{i, \rho}^\pm = \widehat{K}_i^\pm \forall i = 1, \dots, n$. Therefore, by (27), there exists a $\rho > 0$, such that $\forall i = 1, \dots, n$, we have

$$\check{K}_i^- < \widehat{K}_{i, \rho}^- \leq \widehat{K}_{i, \rho}^+ < \check{K}_i^+. \quad (28)$$

Let $\epsilon > 0$ be sufficiently small, such that

$$\check{K}_i^- < \widehat{K}_{i, \rho}^- - \epsilon < \widehat{K}_{i, \rho}^- \leq \widehat{K}_{i, \rho}^+ < \widehat{K}_{i, \rho}^+ + \epsilon < \check{K}_i^+$$

and let us set $\forall i = 1, \dots, n$,

$$\begin{aligned} W_i &= \frac{W_{i_1}}{2 \cos \phi_i} \cos((\arcsin((\widehat{K}_{i, \rho}^+ + \epsilon) d_i \cos \phi_i) - \phi_i) \\ &\quad \vee (\phi_i - \arcsin((\widehat{K}_{i, \rho}^- - \epsilon) d_i \cos \phi_i))). \end{aligned} \quad (29)$$

We now prove (18) and (19) by induction. First of all, let us take $i = 1$; in this case, (15) is exactly (8). If we apply Proposition 1 with $K_1^\pm = \widehat{K}_{1, \rho}^\pm$, noting that \check{K}_0^\pm in Proposition 1 is just $\chi_{d_1, \phi_1}^{-1}(K_1^\pm)$, we get

$$\check{K}_0^\pm = \chi_{d_1, \phi_1}^{-1}(\widehat{K}_{1, \rho}^\pm) = \chi_{d_1, \phi_1}^{-1}(\chi_{d_1, \phi_1}(\widehat{K}_{0, \rho}^\pm \pm \rho)) = K_0^\pm \pm \rho$$

which implies condition (10). Therefore, by Proposition 1, we have

$$\lim_{t \rightarrow +\infty} E_1(t) = 0$$

and

$$0 < W_1 \leq v_1(t) \leq V_1$$

$$K_1^- \leq \check{K}_1^- < \widehat{K}_{1, \rho}^- \leq \kappa_1(t) \leq \widehat{K}_{1, \rho}^+ < \check{K}_1^+ \leq K_1^+.$$

This means that (18) is verified for $i = 1$. Let us suppose by induction that (18) holds for $1 \leq i \leq m-1$ (with $1 < m \leq n$), and let us take,

in Proposition 1, $K_0^\pm = \widehat{K}_{l_m, \rho}^\pm$ and $K_1^\pm = \widehat{K}_{m, \rho}^\pm$. Therefore, \widehat{K}_0^\pm in Proposition 1 is given by

$$\begin{aligned} \widehat{K}_0^\pm &= \chi_{d_m, \phi_m}^{-1}(\widehat{K}_{m, \rho}^\pm) \\ &= \chi_{d_m, \phi_m}^{-1}(\chi_{d_m, \phi_m}(\widehat{K}_{l_m, \rho}^\pm \pm \rho)) = \widehat{K}_{l_m, \rho}^\pm \pm \rho. \end{aligned}$$

Moreover, by (15), (28), and the inductive hypotheses, we have

$$\begin{aligned} &\left. \begin{aligned} \text{if } \phi_m < 0, & \quad -\frac{1}{d_m \cos \phi_m} \\ \text{if } \phi_m \geq 0, & \quad -\frac{1}{d_m} \end{aligned} \right\} \leq \\ K_{l_m}^- &\leq \check{K}_{l_m}^- < \widehat{K}_{l_m, \rho}^- \leq \kappa_{l_m}(t) \leq \widehat{K}_{l_m, \rho}^+ < \check{K}_{l_m}^+ \leq K_{l_m}^+ \\ &\leq \begin{cases} \frac{1}{d_m \cos \phi_m}, & \text{if } \phi_m > 0 \\ \frac{1}{d_m}, & \text{if } \phi_m \leq 0. \end{cases} \end{aligned}$$

Therefore, all the hypotheses of Proposition 1 are verified if we take, in Proposition 1, $K_0^\pm = \widehat{K}_{l_m, \rho}^\pm$, $K_1^\pm = \widehat{K}_{m, \rho}^\pm$, and $\check{K}_0^\pm = \widehat{K}_{l_m, \rho}^\pm \pm \rho$. Then, applying it again, we obtain that (18) holds for $i = m$ and $\lim_{t \rightarrow +\infty} E_m(t) = 0$. Finally, (19) is a straight consequence of Remark 2 applied to each leader-follower unit $(\mathbf{R}_{l_i}, \mathbf{R}_i)$. ■

REFERENCES

- [1] F. Bullo, J. Cortés, and S. Martínez, *Distributed Control of Robotic Networks* (Applied Mathematics Series). Princeton, NJ: Princeton Univ. Press, 2009.
- [2] L. E. Parker, "Multiple mobile Robot systems," in *Handbook of Robotics*, B. Siciliano and O. Khatib, Eds., New York: Springer-Verlag, 2008, ch. 40, pp. 921–941.
- [3] V. Kumar, D. Rus, and G. S. Sukhatme, "Networked robots," in *Handbook of Robotics*, B. Siciliano and O. Khatib, Eds. New York: Springer-Verlag, 2008, ch. 41, pp. 943–958.
- [4] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.
- [5] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.
- [6] H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita, "Distributed memoryless point convergence algorithm for mobile robots with limited visibility," *IEEE Trans. Robot. Autom.*, vol. 15, no. 5, pp. 818–828, Oct. 1999.
- [7] J. Lin, A. S. Morse, and B. D. O. Anderson, "The multi-agent rendezvous problem. Part 1: The synchronous case," *SIAM J. Control Optim.*, vol. 46, no. 6, pp. 2096–2119, 2007.
- [8] J. Cortés, S. Martínez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Trans. Robot. Autom.*, vol. 20, no. 2, pp. 243–255, Apr. 2004.
- [9] J. Meng and M. Egerstedt, "Distributed coordination control of multiagent systems while preserving connectedness," *IEEE Trans. Robot.*, vol. 23, no. 4, pp. 693–703, Aug. 2007.
- [10] D. V. Dimarogonas and K. J. Kyriakopoulos, "Connectedness preserving distributed swarm aggregation for multiple kinematic robots," *IEEE Trans. Robot.*, vol. 24, no. 5, pp. 1213–1223, Oct. 2008.
- [11] K. H. Tan and M. A. Lewis, "High precision formation control of mobile robots using virtual structures," *Auton. Robots*, vol. 4, no. 4, pp. 387–403, 1997.
- [12] T. Balch and R. C. Arkin, "Behavior-based formation control for multi-robot teams," *IEEE Trans. Robot. Autom.*, vol. 14, no. 6, pp. 926–939, Dec. 1998.
- [13] M. Egerstedt and X. Hu, "Formation constrained multi-agent control," *IEEE Trans. Autom. Control*, vol. 17, no. 6, pp. 947–951, Dec. 2001.
- [14] P. Tabuada, G. J. Pappas, and P. Lima, "Motion feasibility of multi-agent formations," *IEEE Trans. Robot.*, vol. 21, no. 3, pp. 387–392, Jun. 2005.
- [15] W. Dong and J. A. Farrell, "Cooperative control of multiple nonholonomic mobile agents," *IEEE Trans. Autom. Control*, vol. 53, no. 6, pp. 1434–1448, Jul. 2008.

- [16] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1465–1476, Sep. 2004.
- [17] Z. Lin, B. A. Francis, and M. Maggiore, "Necessary and sufficient graphical conditions for formation control of unicycles," *IEEE Trans. Autom. Control*, vol. 50, no. 1, pp. 121–127, Jan. 2005.
- [18] T. I. Fossen, *Guidance and Control of Ocean Vehicles*. New York: Wiley, 1994.
- [19] D. B. Edwards, T. A. Bean, D. L. Odell, and M. J. Anderson, "A leader-follower algorithm for multiple AUV formations," in *Proc. IEEE/OES Autom. Underwater Vehicles*, 2004, pp. 40–46.
- [20] T. J. Koo and S. M. Shahruz, "Formation of a group of unmanned aerial vehicles (UAVs)," in *Proc. Amer. Control Conf.*, 2001, pp. 69–74.
- [21] L. E. Buzogany, M. Pachter, and J. J. D'Azzo, "Automated control of aircraft in formation flight," in *Proc. AIAA Guid., Navigat., Control Conf.*, 1993, pp. 1349–1370.
- [22] H. Schaub, S. R. Vadali, J. L. Junkins, and K. T. Alfriend, "Spacecraft formation flying control using mean orbit elements," *J. Astronaut. Sci.*, vol. 48, no. 1, pp. 69–87, 2000.
- [23] R. Burns, C. A. McLaughlin, J. Leitner, and M. Martin, "TechSat21: Formation design, control and simulation," in *Proc. IEEE Aerosp. Conf.*, 2000, vol. 7, pp. 19–25.
- [24] A. K. Das, R. Fierro, V. Kumar, J. P. Ostrowsky, J. Spletzer, and C. Taylor, "A vision-based formation control framework," *IEEE Trans. Robot. Autom.*, vol. 18, no. 5, pp. 813–825, Oct. 2002.
- [25] H. G. Tanner, G. J. Pappas, and V. Kumar, "Leader-to-formation stability," *IEEE Trans. Robot. Autom.*, vol. 20, no. 3, pp. 443–455, Jun. 2004.
- [26] F. Morbidi, G. L. Mariottini, and D. Prattichizzo, "Vision-based range estimation via Immersion and Invariance for robot formation control," in *Proc. IEEE Int. Conf. Robot. Autom.*, 2008, pp. 504–509.
- [27] L. Consolini, F. Morbidi, D. Prattichizzo, and M. Tosques, "Leader-follower formation control of nonholonomic mobile robots with input constraints," *Automatica*, vol. 44, no. 5, pp. 1343–1349, 2008.

Shortest Paths to Obstacles for a Polygonal Dubins Car

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Abstract—In this paper, we characterize the time-optimal trajectories leading a Dubins car in collision with the obstacles in its workspace. Due to the constant velocity constraint characterizing the Dubins car model, these trajectories form a sufficient set of shortest paths between any robot configuration and the obstacles in the environment. Based on these paths, we define and give the algorithm for computing a distance function that takes into account the nonholonomic constraints and captures the nonsymmetric nature of the system. The developments presented here assume that the obstacles and the robot are polygons although the methodology can be applied to different shapes.

Index Terms—Dubins car, Pontryagin's maximum principle, shortest paths.

I. INTRODUCTION

In his pioneering work [1], Dubins determined the continuously differentiable curves of minimal length between any two points in the plane with assigned initial and final tangent and subject to curvature

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constraints. These curves are time-optimal trajectories for a wheeled mobile robot with unitary speed and bounded steering velocity. For this reason, a vehicle with constant driving velocity is known in the literature as the Dubins car. The interest arisen by this system is due to both the theoretical challenges that it provides and its practical relevance in modeling the kinematics of road vehicles, aircrafts cruising at constant altitude, or sea vessels.

The difficulty of planning for a wheeled mobile robot derives from the pure rolling constraint of the wheels that prevents the vehicle to move instantaneously toward certain directions. As a consequence, not all the paths in the configuration space are feasible for this robot, and the Euclidean metric is not appropriate for determining the distance to obstacles populating the robot environment. In a previous work [2], we have defined and showed how to compute a distance between a pointwise robot and the obstacles in its environment by taking into account the nonholonomic constraints acting on the vehicle. In this paper, we consider both the nonholonomic constraint and the shape of the robot. Specifically, we define and compute a distance in the robot configuration space that takes into account the nonholonomic constraints of the vehicle. The computation does not require the explicit representation of the configuration space obstacles and is applicable to robots and obstacles that can be modeled as planar polygons. This distance function is particularly relevant in the context of motion planners relying on obstacle distance computation, like skeletonization, potential field methods, and even sampling-based motion planners. See, e.g., [3] for an interesting discussion on the use of the obstacle distance information in planning algorithms. In particular, the correct metric information provided by the distance developed in this paper can improve the efficiency of sampling-based methods: by computing the distance to the obstacles in the environment, it is possible to adjust the resolution at which configurations along local paths are checked for collision, in the line of [4]. This avoids the difficult resolution tuning step and provides an exact solution to the collision check test for local paths.

Our work relies on Dubins' results but adopts the optimal control approach proposed in [5] and [6]. In particular, the study of transversality conditions allows selecting a sufficient family of time-optimal trajectories whose length will determine the distance to the obstacles. A preliminary version of this paper has been presented in [7]. Here, we present a revised work extended with detailed proofs and formal equations for distance computation. The paper is organized as follows. In Section II, we summarize the basic results on time-optimal trajectories for the Dubins car. Due to the constant velocity constraint, these trajectories also minimize the path length. This is exactly the property that we exploit in defining the distance function in Section III where we also give the necessary conditions satisfied by the shortest paths bringing the robot in collision with obstacles. The distance function is based on the length of the shortest paths belonging to a sufficient family. In Section IV, we show how to reduce the cardinality of this set of paths to lower the distance computation complexity, and in Section V, we provide the formal system of equations needed for implementation. These are used in Section VI to obtain the isodistance curves on the xy plane for two robots with different shapes.

II. BACKGROUND MATERIAL

The kinematic model of the Dubins car considered in this paper is

$$\dot{\xi}(t) = f(\xi(t), u(t)) = g_1(\xi(t)) + g_2(\xi(t))u(t) \quad (1)$$

where $\xi \in R^2 \times S^1$ denotes the configuration of the vehicle, given by the position (x_r, y_r) of a reference point P , chosen as the mid point of the rear wheels' axle, and by the orientation θ of the vehicle w.r.t. the abscissa axis of a fixed reference frame. The drift and the input