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Brief paper

Leader–follower formation control of nonholonomic mobile robots with input constraints[☆]

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Abstract

The paper deals with leader–follower formations of nonholonomic mobile robots, introducing a formation control strategy alternative to those existing in the literature. Robots' control inputs are forced to satisfy suitable constraints that restrict the set of leader possible paths and admissible positions of the follower with respect to the leader. A peculiar characteristic of the proposed strategy is that the follower position is not rigidly fixed with respect to the leader but varies in proper circle arcs centered in the leader reference frame.

Keywords: Formations control; Autonomous mobile robots; Input constraints; Geometric approach; Stabilization

1. Introduction

In the last decade formation control became one of the leading research areas in mobile robotics. By formation control we simply mean the problem of controlling the relative position and orientation of the robots in a group while allowing the group to move as a whole. Different robot formation typologies have been studied in the literature: ground vehicles (Das et al., 2002; Fax & Murray, 2004; Ghabcheloo, Pascoal, Silvestre, & Kaminer, 2006; Lin, Francis, & Maggiore, 2005; Marshall, Broucke, & Francis, 2004), unmanned aerial vehicles (UAVs) (Koo & Shahruz, 2001; Singh, Chandler, Schumacher, Banda, & Pachter, 2000), aircraft (Fierro, Belta, Desai, & Kumar, 2001; Giulietti, Pollini, & Innocenti, 2000), surface and underwater autonomous vehicles (AUVs) (Edwards, Bean, Odell, & Anderson, 2004; Skjetne, Moi, & Fossen, 2002). Three main approaches have been proposed to tackle the robot formation control problem: behavior based, virtual structure and leader following. In the behavior-based approach (Balch & Arkin, 1998; Lawton, Beard, & Young, 2003) several desired behaviors (e.g. collision avoidance, formation keeping, target seeking) are prescribed for each robot. The resulting action of each robot is derived by weighing the relative importance of each behavior. The main problem of this approach is that the mathematical formalization is difficult and consequently it is not easy to guarantee the convergence of the formation to a desired configuration. The virtual structure approach (Do & Pan, 2007; Lewis & Tan, 1997) considers the formation as a single virtual rigid structure so that the behavior of the robotic system is similar to that of a physical object. Desired trajectories are not assigned to each single robot but to the entire formation as a whole. The behavior of the formation in this case is exactly predictable but a large inter-robot communication bandwidth is required. In the leader–follower approach a robot of the formation, designed as the leader, moves along a predefined trajectory while the other robots, the followers, are to maintain a desired distance and orientation to the leader (Das et al., 2002). Perhaps the main criticism to the leader–follower approach is that it depends heavily on the leader for achieving the goal and over-reliance on a single agent in the formation may be undesirable, especially in adverse conditions (Fax &

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Murray, 2004). Nevertheless leader–follower architectures are particularly appreciated for their simplicity and scalability.

The leader–follower formation control of nonholonomic mobile robots with bounded control inputs is the subject of this paper. We propose a leader–follower setup alternative to those existing in the literature (Mariottini, Morbidi, Prattichizzo, Pappas, & Daniilidis, 2007), assuming that the desired angle between the leader and the follower is measured in the follower frame instead of the leader frame. As shown in Consolini, Morbidi, Prattichizzo, and Tosques (2006), this approach guarantees lower control effort and smoother trajectories for the follower than the usual leader–follower approaches defined on the leader reference frame (Das et al., 2002). According to the proposed setup we find suitable conditions on leader velocity and trajectory curvature ensuring the follower gets into and keeps the formation while satisfying its own velocity constraints. As pointed out in Remark 1, from a geometric point of view the follower position is not fixed with respect to the leader reference frame but varies in suitable circle arcs centered in the leader reference frame. The main contributions of this paper are stated in two theorems. A first theorem gives a sufficient and necessary condition on leader and follower velocity bounds for the existence of a control law that allows the follower to maintain the formation independently from the trajectory of the leader. A second theorem provides a sufficient condition and a control law for the follower to asymptotically reach the formation for any initial condition and any leader motion, while still respecting the bounds.

Notation. The following notation is used in the paper: $\mathbb{R}^+ = \{t \in \mathbb{R} | t \geq 0\}$; $\forall\, a, b \in \mathbb{R}$, $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$; $\forall\, t > 0$, $\operatorname{sign}(t) = 1$; $\operatorname{sign}(0) = 0$; $\forall\, t < 0$, $\operatorname{sign}(t) = -1$; $\forall\, x, y \in \mathbb{R}^n$ $(n \geq 1)$, $\langle x, y\rangle = \sum_{i=1}^n x_i\, y_i$, $\|x\| = \sqrt{\langle x, x\rangle}$; $\forall\, \theta \in \mathbb{R}$, $\tau(\theta) = (\cos\theta, \sin\theta)^{\mathrm{T}}$, $\nu(\theta) = (-\sin\theta, \cos\theta)^{\mathrm{T}}$; $\forall\, x \in \mathbb{R}^2 \backslash \{0\}$, $\arg(x) = \theta$ where $\theta \in [0, 2\pi)$ and $x = \|x\|\, \tau(\theta)$; $\forall\, w \in \mathbb{R}^2$, $w^{\perp} = \|w\|\, \nu(\arg(w))$.

2. Basic definitions

Consider the following definition of robot as a velocity controlled unicycle model.

Definition 1. Let $\mathbf{R} = (x, y, \theta)^{\mathrm{T}} \in \mathcal{C}^1([0, +\infty), \mathbb{R}^3)$. $\mathbf{R}$ is called a robot with initial condition $\overline{\mathbf{R}} \in \mathbb{R}^3$ (and control $(v, \omega)^{\mathrm{T}} \in \mathcal{C}^0([0, +\infty), \mathbb{R}^2)$) if the following system is verified

$$\begin{cases} \dot{x} = v\cos\theta, \qquad \dot{y} = v\sin\theta, \qquad \dot{\theta} = \omega \\ (x(0), y(0), \theta(0))^{\mathrm{T}} = \overline{\mathbf{R}}. \end{cases} \tag{1}$$

If $v(t) \neq 0$, we set $\kappa(t) = \omega(t)/v(t)$ which is the (scalar) curvature of the path followed by the robot at time t. Denote by $P(t) = (x(t), y(t))^{\mathrm{T}}$ the position of $\mathbf{R}$ at time t, $\theta(t)$ its heading, $\tau(\theta(t))$ the normalized velocity vector and $\nu(\theta(t))$ the normalized vector orthogonal to $\tau(\theta(t))$; then $\{\tau(\theta(t)), \nu(\theta(t))\}$ represents the robot reference frame at time t.

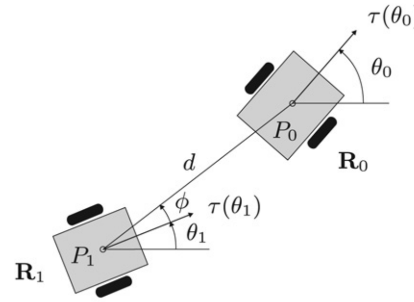

Fig. 1. (d, ϕ)-formation.

Definition 2. Let $(V, K^-, K^+) \in \mathbb{R}^3$ and $\mathbf{R}$ be a robot. We say that $\mathbf{R}$ satisfies the trajectory constraint (V, K^-, K^+) if $\forall\, t \geq 0$

$$0 < v(t) \leq V, \qquad K^- \leq \kappa(t) \leq K^+.$$

Assumption 1 (*Physical constraint*). Let V_p, K_p^-, K_p^+ be three constants. All along the paper, we will suppose that every robot satisfies the trajectory constraint (V_p, K_p^-, K_p^+).

This constraint is assumed to be a mechanical limitation common to all the robots.

The following definition introduces the notion of leader–follower formation used in the paper.

Definition 3. Let $d > 0$, $\phi : |\phi| < \frac{\pi}{2}$ and let $\mathbf{R}_0 = (P_0^{\mathrm{T}}, \theta_0)^{\mathrm{T}}$, $\mathbf{R}_1 = (P_1^{\mathrm{T}}, \theta_1)^{\mathrm{T}}$ be two robots. We say that $\mathbf{R}_0$ and $\mathbf{R}_1$ are in (d, ϕ)-formation with leader $\mathbf{R}_0$ at time t, if

$$P_0(t) = P_1(t) + d\tau(\theta_1(t) + \phi), \tag{2}$$

and, simply, that $\mathbf{R}_0$ and $\mathbf{R}_1$ are in (d, ϕ)-formation with leader $\mathbf{R}_0$, if (2) holds for all $t \geq 0$. Moreover we say that $\mathbf{R}_0$ and $\mathbf{R}_1$ are *asymptotically* in (d, ϕ)-formation with leader $\mathbf{R}_0$ if

$$\lim_{t\to\infty} P_0(t) - (P_1(t) + d\tau(\theta_1(t) + \phi)) = 0.$$

With reference to Fig. 1, Definition 3 states that two robots $\mathbf{R}_0$, $\mathbf{R}_1$ are in (d, ϕ)-formation with leader $\mathbf{R}_0$ if the position P_1 of the follower $\mathbf{R}_1$ is always at distance d from the position P_0 of the leader $\mathbf{R}_0$ and the angle between vectors $\tau(\theta_1)$ and $P_0 - P_1$ is constantly equal to ϕ, that is the position P_0 remains fixed with respect to the follower reference frame $\{\tau(\theta_1), \nu(\theta_1)\}$. An equivalent way to state Definition 3 is the following: set the error vector,

$$E(t) = P_0(t) - (P_1(t) + d\tau(\theta_1(t) + \phi)), \tag{3}$$

then $\mathbf{R}_0$ and $\mathbf{R}_1$ are in (d, ϕ)-formation (asymptotically) if and only if $E(t) = 0$, $\forall\, t \geq 0$ (respectively, $\lim_{t\to\infty} E(t) = 0$).

3. Characterization of a necessary and sufficient condition for the leader–follower formation

Problem 1. Set $d > 0$, $\phi : |\phi| < \frac{\pi}{2}$. Let $\mathbf{R}_0$ be a robot satisfying the trajectory constraint (V_0, K_0^-, K_0^+) and $\mathbf{R}_1$ be another robot. Find necessary and sufficient conditions on V_0, K_0^-, K_0^+ such that there exist controls v_1, ω_1 and suitable

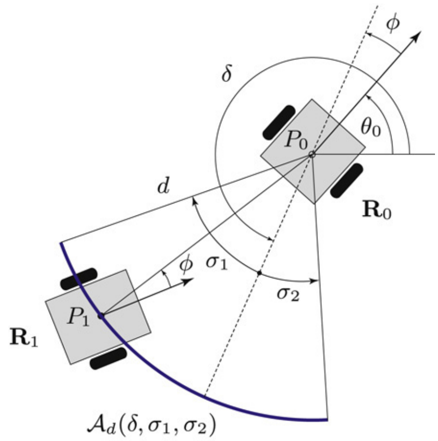

Fig. 2. The arc of circle $\mathcal{A}_d(\delta, \sigma_1, \sigma_2)$.

initial conditions for $\mathbf{R}_1$ such that $\mathbf{R}_0$ and $\mathbf{R}_1$ are in (d, ϕ)-formation with leader $\mathbf{R}_0$ (remark that v_1 and ω_1 must be such that Assumption 1 is satisfied).

The following theorem gives an answer to Problem 1.

Theorem 1. *Set* $d > 0$, $\phi : |\phi| < \frac{\pi}{2}$.

(1) *The following properties are equivalent:*

(A) *For any robot* $\mathbf{R}_0 = (P_0^{\mathrm{T}},\ \theta_0)^{\mathrm{T}}$ *with initial condition* $\overline{\mathbf{R}}_0$, *satisfying the trajectory constraint* (V_0, K_0^-, K_0^+), *there exists an initial condition* $\overline{\mathbf{R}}_1$ *and controls* v_1, ω_1 *for the robot* $\mathbf{R}_1 = (P_1^{\mathrm{T}},\ \theta_1)^{\mathrm{T}}$ *such that* $\mathbf{R}_0$ *and* $\mathbf{R}_1$ *are in* (d, ϕ)*-formation with leader* $\mathbf{R}_0$.

(B) *The following properties hold:*

$$\begin{aligned} -\frac{1}{d} \le K_0^- \le K_0^+ \le \frac{1}{d\cos\phi}, &\quad \text{if } \phi \ge 0 \\ -\frac{1}{d\cos\phi} \le K_0^- \le K_0^+ \le \frac{1}{d}, &\quad \text{if } \phi < 0 \end{aligned} \tag{4}$$

$$\widetilde{K}_0^- \le K_0^- \le K_0^+ \le \widetilde{K}_0^+ \tag{5}$$

$$V_0 \cos(0 \wedge (\arcsin(K_0^+ d\cos\phi) - \phi) \wedge (\phi - \arcsin(K_0^- d\cos\phi))) \le V_p \cos\phi \tag{6}$$

where

$$\widetilde{K}_0^{\pm} = (\,\text{sign } K_p^{\pm})(((K_p^{\pm})^{-1} - d\sin\phi)^2 + d^2\cos^2\phi)^{-\frac{1}{2}}. \tag{7}$$

(2) *If* **(B)** *holds, for any robot* $\mathbf{R}_0$ *satisfying the trajectory constraint* (V_0, K_0^-, K_0^+) *and for any robot* $\mathbf{R}_1$ *which is in* (d, ϕ)*-formation at time* $t = 0$ *and* $\arcsin(K_0^- d\cos\phi) \le \theta_0(0) - \theta_1(0) \le \arcsin(K_0^+ d\cos\phi)$, *with controls,*

$$v_1 = v_0(t)\frac{\cos(\theta_0 - \theta_1 - \phi)}{\cos\phi}, \quad \omega_1 = v_0(t)\frac{\sin(\theta_0 - \theta_1)}{d\cos\phi} \tag{8}$$

then $\mathbf{R}_0$ *and* $\mathbf{R}_1$ *are in* (d, ϕ)*-formation and* $\forall\, t \ge 0$,

$$\arcsin(K_0^- d\cos\phi) \le \theta_0(t) - \theta_1(t) \le \arcsin(K_0^+ d\cos\phi). \tag{9}$$

Remark 1. Property (9) shows the connection between the relative heading angle $\theta_0 - \theta_1$, the curvature of the path followed by the leader $\mathbf{R}_0$, the distance d and the visual angle ϕ. Moreover this implies the following geometric property. Since robots $\mathbf{R}_0$, $\mathbf{R}_1$ are in (d, ϕ)-formation, $P_0(t) - P_1(t) = d\ \tau(\theta_1(t) + \phi) = d\ \tau((\theta_0(t) + \phi) + (\theta_1(t) - \theta_0(t)))$, therefore, defining, for any $\sigma_1 \le \sigma_2$, $\mathcal{A}_d(\delta(t), \sigma_1, \sigma_2) = d\tau([\delta(t) + \sigma_1, \delta(t) + \sigma_2])$ as the arc of circle centered in the origin, radius d, angle of the reference axis $\delta(t)$ and aperture $(\sigma_2 - \sigma_1)$, by (9), $P_0(t) - P_1(t) \in \mathcal{A}_d(\theta_0(t) + \phi, -\arcsin(K_0^+ d\cos\phi), -\arcsin(K_0^- d\cos\phi))$ that is

$$\begin{aligned} P_1(t) \in P_0(t) + \mathcal{A}_d(\theta_0(t) + \phi + \pi, \\ -\arcsin(K_0^+ d\cos\phi), -\arcsin(K_0^- d\cos\phi)) \end{aligned}$$

(see Fig. 2 where $\sigma_1 = -\arcsin(K_0^+ d\cos\phi)$, $\sigma_2 = -\arcsin(K_0^- d\cos\phi)$ and $\delta(t) = \theta_0(t) + \phi + \pi$, in the case $K_0^- < 0 < K_0^+$). This shows that, differently from Das et al. (2002), the followers are not rigidly disposed with respect to the leader reference frame, but their relative positions vary in time in suitable circle arcs and only these remain stable with respect to the leader reference frame

Proof. (A) $\Rightarrow$ **(B)**. Suppose that $\mathbf{R}_0$ and $\mathbf{R}_1$ are in (d, ϕ)-formation, the trajectory constraint

$$0 < v_0(t) \le V_0, \qquad K_0^- \le \kappa_0(t) \le K_0^+, \quad \forall t \ge 0 \tag{10}$$

is verified and (by Assumption 1) the physical constraint

$$0 < v_1(t) \le V_p, \qquad K_p^- \le \kappa_1(t) \le K_p^+, \quad \forall t \ge 0 \tag{11}$$

holds for robot $\mathbf{R}_1$. Since $E(t) = P_0(t) - (P_1(t) + d\ \tau(\theta_1(t) + \phi)) = 0$, differentiating it, we get that $\dot{P}_0 = \dot{P}_1 + d\ \dot{\theta}_1 \nu(\theta_1 + \phi)$, that is $v_0 \tau(\theta_0) = v_1\ \tau(\theta_1) + d\ \omega_1\ \nu(\theta_1 + \phi)$ which implies, multiplying by the rotation matrix $\mathcal{R}(-\theta_1) = (\tau(-\theta_1),\ \nu(-\theta_1))$, that $v_0\ \tau(\theta_0 - \theta_1) = v_1\ (1,\ 0)^{\mathrm{T}} + d\omega_1\ \nu(\phi)$. Therefore it has to be

$$\omega_1 = v_0 \frac{\sin(\theta_0 - \theta_1)}{d\cos\phi}, \tag{12}$$

$$\begin{aligned} v_1 &= v_0 \left[\cos(\theta_0 - \theta_1) + \frac{\sin(\theta_0 - \theta_1)\sin\phi}{\cos\phi}\right] \\ &= v_0 \frac{\cos(\theta_0 - \theta_1 - \phi)}{\cos\phi}. \end{aligned} \tag{13}$$

Then (8) is true and

$$|\theta_0(t) - \theta_1(t)| < \pi/2, \quad \forall\, t \ge 0, \tag{14}$$

since v_1 has to be greater than zero by (11). Furthermore, setting $\beta(t) = \theta_0(t) - \theta_1(t)$, on $[0, \ +\infty)$

$$\dot{\beta} = \omega_0 - \omega_1 = \frac{v_0}{d\cos\phi}\ (\kappa_0\ d\cos\phi - \sin\beta). \tag{15}$$

Let $\beta \in \mathcal{C}^1([0, +\infty), \mathbb{R})$ be a solution of (15), then it is easy to verify that the following properties hold:

(a) if $|\kappa_0(t)\ d\ \cos\phi| \ge c > 1$, $\forall\, t \ge 0$ then

$$\lim_{t\to+\infty} |\beta(t)| = +\infty;$$

(b) let K be a constant such that $|K|\, d\cos\phi \le 1$ if $\forall\, t \ge 0$, $\kappa_0(t) = K < 0\ (> 0)$ and $-\frac{\pi}{2} \le \beta(0) \le \pi, (-\pi \le \beta(0) \le \frac{\pi}{2}$, respectively), then $\lim_{t\to+\infty} \beta(t) = \arcsin(K d\cos\phi)$;

(c) if $-\frac{1}{d\cos\phi} \le K_0^- \le K_0^+ \le \frac{1}{d\cos\phi}$, $\arcsin(K_0^- d\cos\phi) \le (<) \beta(0) \le (<) \arcsin(K_0^+ d\cos\phi)$, then $\arcsin(K_0^- d\cos\phi) \le (<) \beta(t) \le (<) \arcsin(K_0^+ d\cos\phi)$, $\forall\, t \ge 0$, where the notation $\le (<)$ means that the property holds also when all inequalities "$\le$" are replaced with "$<$".

To verify (4) suppose that $\phi > 0$ (the reasoning is analogous for $\phi \le 0$). By contradiction, suppose that (4) is false, then at least one of the following two inequalities must be true: $K_0^+ > \frac{1}{d\cos\phi}$, $K_0^- < -\frac{1}{d}$. Suppose that $K_0^+ > \frac{1}{d\cos\phi}$, and that $\mathbf{R}_0$ follows a circle of curvature K_0^+ with constant velocity V_0. Then hypothesis **(a)** holds and $\lim_{t\to\infty} |\beta(t)| = +\infty$, therefore β is unbounded which is impossible by (14). Suppose that $K_0^- < -\frac{1}{d}$, if ϕ is such that $K_0^- d\cos\phi < -1$, then, by **(a)** as before, a contradiction follows; suppose that $K_0^- d\cos\phi \ge -1$, since $-\frac{\pi}{2} \le \beta(0) \le \pi$, being $|\beta(0) - \phi| < \frac{\pi}{2}$ by (14) and $0 < \phi < \frac{\pi}{2}$ we get that $\lim_{t\to+\infty} \beta(t) = \arcsin(K_0^- d\cos\phi)$, by property **(b)** applied with $K = K_0^- < 0$. Therefore $\lim_{t\to\infty} v_1(t) = V_0(\sqrt{1 - (K_0^- d\cos\phi)^2} + K_0^- d\sin\phi)$, which implies $\lim_{t\to\infty} v_1(t) < 0$, since $K_0^- < 0$ and $1 < (K_0^- d)^2$ which contradicts (11), therefore (4) holds. Suppose now that (5) does not hold, for instance that

$$K_0^+ > \widetilde{K}_0^+ = \frac{\operatorname{sign} K_p^+}{\sqrt{((K_p^+)^{-1} - d\sin\phi)^2 + d^2\cos^2\phi}}. \tag{16}$$

Suppose for example that $K_0^+ \ge K_0^- \ge 0$ (the reasoning for the other case is analogous), then (16) implies that $K_p^+ < (\sqrt{(K_0^+)^{-2} - d^2\cos^2\phi} + d\sin\phi)^{-1}$. Suppose that $\mathbf{R}_0$ follows a circle of curvature K_0^+ with constant velocity V_0. Then by the necessity hypothesis there exist initial conditions such that $\mathbf{R}_0$ and $\mathbf{R}_1$ are in (d,ϕ)-formation and it has to be $\lim_{t\to\infty} \sin\beta(t) = K_0^+ d\cos\phi$, since β is such that $\dot\beta = \frac{V_0}{d\cos\phi}(K_0^+ d\cos\phi - \sin\beta)$ and $K_0^+ d\cos\phi \le 1$, by (4). Therefore, by (8): $\lim_{t\to+\infty} \kappa_1(t) = \lim_{t\to+\infty} \frac{\omega_1(t)}{v_1(t)} = \lim_{t\to\infty} \sin\beta(t)(d(\cos\beta(t)\cos\phi + \sin\beta(t)\sin\phi))^{-1} = \lim_{t\to\infty} \frac{1}{d}(\operatorname{sign}\beta(t)\sqrt{\frac{1}{\sin^2\beta(t)} - 1}\ \cos\phi + \sin\phi)^{-1} = (\sqrt{(K_0^+)^{-2} - d^2\cos^2\phi} + d\sin\phi)^{-1} > K_p^+$ which is impossible by (11). By contradiction suppose that (6) is false and set for simplicity $\alpha^\pm = \arcsin(K_0^\pm d\cos\phi)$ (remark that $|K_0^\pm d\cos\phi| \le 1$ by (4)). Suppose first of all that $0 \wedge (\alpha^+ - \phi) \wedge (\phi - \alpha^-) = \alpha^+ - \phi$, analogously we reason if $0 \wedge (\alpha^+ - \phi) \wedge (\phi - \alpha^-) = \phi - \alpha^-$. Let $\mathbf{R}_0$ be the robot which follows a circle with curvature K and velocity V_0. By hypothesis there exists an initial condition $\overline{\mathbf{R}}_1$ and controls v_1, w_1 such that robot $\mathbf{R}_1$ is in (d,ϕ)-formation with leader $\mathbf{R}_0$. If $\beta(0) = \theta_0(0) - \theta_1(0)$ is different from $\pi - \alpha^+$ then set $K = K_0^+$, $\lim_{t\to+\infty} \beta(t) = \alpha^+$ and $\lim_{t\to+\infty} v_1(t) = V_0 \frac{\cos(\alpha^+ - \phi)}{\cos\phi} > V_p$, therefore the first inequality of (11) is false. If $\beta(0) = \theta_0(0) - \theta_1(0) = \pi - \alpha^+$, take $K = K_0^+ - \epsilon$ with ϵ chosen such that $\frac{V_0 \cos(\arcsin((K_0^+ - \epsilon)\ d\cos\phi) - \phi)}{\cos\phi} > V_p$. Since $\beta(0)$ is now different from $\pi - \alpha^+$ as before, $\lim_{t\to+\infty} v_1(t) > V_p$, which contradicts (11). If $0 \wedge (\alpha^+ - \phi) \wedge (\phi - \alpha^-) = 0$ then take $K = \frac{\tan\phi}{d}$ (which is the solution of $\arcsin(K d\cos\phi) - \phi = 0$); since $\alpha^- \le \phi \le \alpha^+$, it follows that $K_0^- \le K \le K_0^+$ and reasoning as before, we get a contradiction. Therefore (6) holds. Finally to verify (9), remark that if $\mathbf{R}_0$ and $\mathbf{R}_1$ are in (d,ϕ)-formation then it is necessary that $-\frac{1}{d\cos\phi} \le K_0^- \le K_0^+ \le \frac{1}{d\cos\phi}$ (by (4)), therefore (9) follows from property **(c)**.

(B) $\Rightarrow$ **(A)**. Let $\mathbf{R}_0$ be a robot which verifies the trajectory constraint (10) with V_0, K_0^-, K_0^+ satisfying conditions (4), (5) and (6). The aim is to show that there exist initial conditions such that $\mathbf{R}_1$ with controls v_1, ω_1 given by (8) is in (d,ϕ)-formation with $\mathbf{R}_0$ and constraint (11) is verified. In fact if $\mathbf{R}_1$ has the controls v_1 and ω_1, by the definition of $E(t)$, given by (3): $\dot E = v_0\ \tau(\theta_0) - (v_1\ \tau(\theta_1) + d\nu(\theta_1 + \phi)\ \omega_1) = v_0\,\tau(\theta_0) - (v_0 \frac{\cos(\theta_0 - \theta_1 - \phi)}{\cos\phi}\ \tau(\theta_1) + d\ \nu(\theta_1 + \phi)\ v_0 \frac{\sin(\theta_0 - \theta_1)}{d\cos\phi}) = \frac{v_0}{\cos\phi}\ \mathcal{R}(\theta_0)[(\cos\phi,\ 0)^{\mathrm{T}} - \sin(\theta_0 - \theta_1)\ \nu(\theta_1 - \theta_0 + \phi) - \cos(\theta_0 - \theta_1 - \phi)\tau(\theta_1 - \theta_0)] = 0$, with $\mathcal{R}(\theta_0) = (\tau(\theta_0),\ \nu(\theta_0))$. Choose the initial condition of the follower $\overline{\mathbf{R}}_1$ such that $\mathbf{R}_0$ and $\mathbf{R}_1$ are in (d,ϕ)-formation at time $t = 0$ and $\arcsin(K_0^- d\cos\phi) < \theta_0(0) - \theta_1(0) < \arcsin(K_0^+ d\cos\phi)$, recall that $|K_0^\pm| d\cos\phi \le 1$ by (4). By **(c)**, $\forall\, t \ge 0$

$$\arcsin(K_0^- d\cos\phi) < \theta_0(t) - \theta_1(t) < \arcsin(K_0^+ d\cos\phi). \tag{17}$$

Consider $\phi \ge 0$ (the case $\phi < 0$ is analogous), then $\theta_0 - \theta_1 - \phi < \arcsin(K_0^+ d\cos\phi) - \phi \le \arcsin(1) - \phi \le \frac{\pi}{2}$ and $\theta_0 - \theta_1 - \phi > \arcsin(K_0^- d\cos\phi) - \phi \ge -\arcsin(\cos\phi) - \phi = -\frac{\pi}{2}$, being $-1 \le K_0^- d$, $K_0^+ d\cos\phi \le 1$, therefore $v_1 = v_0 \frac{\cos(\theta_0 - \theta_1 - \phi)}{\cos\phi} > 0$. Moreover from (17) and (6) it follows directly that $v_1(t) \le V_p$. Finally the curvature of the path followed by $\mathbf{R}_1$ is given by $\kappa_1(t) = \frac{\omega_1(t)}{v_1(t)} = \frac{\sin\beta}{d\cos(\beta - \phi)} = \frac{1}{d(\cot\beta\cos\phi + \sin\phi)}$. Being the function

$$f(\beta) = \begin{cases} (d(\cot\beta\cos\phi + \sin\phi))^{-1} & \text{if } \beta \ne 0, |\beta| \le \dfrac{\pi}{2} \\ 0 & \text{if } \beta = 0 \end{cases}$$

monotone increasing, it follows by (5) that

$$\begin{aligned} &\operatorname{sign} \widetilde{K}_0^- \left(d\ (\cot(\arcsin(d\ \widetilde{K}_1^- \cos\phi))\cos\phi + \sin\phi)\right)^{-1} \\ &\quad\le \frac{\omega_1(t)}{v_1(t)} \le \operatorname{sign} \widetilde{K}_0^+ (d\ (\cot(\arcsin(d\widetilde{K}_1^+ \cos\phi))\cos\phi \\ &\qquad + \sin\phi))^{-1} \end{aligned}$$

therefore

$$\begin{aligned} &\operatorname{sign} \widetilde{K}_0^- (\sqrt{(\widetilde{K}_0^-)^{-2} - d^2\cos^2\phi} + d\ \sin\phi)^{-1} \le \frac{\omega_1(t)}{v_1(t)} \\ &\quad\le \operatorname{sign} \widetilde{K}_0^+ (\sqrt{(\widetilde{K}_0^+)^{-2} - d^2\cos^2\phi}\ + d\sin\phi)^{-1} \end{aligned}$$

which implies by (7) that $K_p^- \le \frac{\omega_1(t)}{v_1(t)} \le K_p^+$.

Therefore all the constraints are satisfied. □

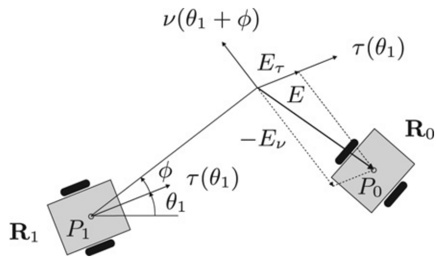

Fig. 3. Error decomposition, where $E_\tau = \frac{\langle E,\, \tau(\theta_1+\phi)\rangle}{\cos\phi}$ and $E_\nu = \frac{\langle E,\, \nu(\theta_1)\rangle}{\cos\phi}$.

4. Stabilization of the leader–follower formation

Problem 2. Set $d > 0$, ϕ: $|\phi| < \frac{\pi}{2}$. Let $\mathbf{R}_0 = (P_0^{\mathrm{T}},\ \theta_0)^{\mathrm{T}}$ be a robot satisfying the trajectory constraint (V_0, K_0^-, K_0^+) and $\mathbf{R}_1$ be another robot. Find sufficient conditions on V_0, K_0^-, K_0^+ that guarantee the existence of controls v_1, ω_1 such that for any initial condition $\overline{\mathbf{R}}_1$, $\mathbf{R}_0$ and $\mathbf{R}_1$ are asymptotically in (d, ϕ)-formation with leader $\mathbf{R}_0$ (remark that v_1 and ω_1 must be such that Assumption 1 is satisfied).

The proposed control strategy consists of two steps. In the first step the follower moves with maximum linear and angular velocities until its direction is sufficiently close to that of the leader in order to satisfy the condition $\alpha^- \le \theta_0 - \theta_1 \le \alpha^+$, where α^- and α^+ are related to the maximum admissible curvatures K_0^-, K_0^+ of the path followed by the leader. In the second step, the follower performs the control defined in the previous section with an added stabilizing term in order to reduce the error asymptotically to zero. The stabilizing term is chosen accurately in order to satisfy constraint (11).

The basic geometric idea under the stabilization is the following: the error vector E is decomposed with respect to vectors $\tau(\theta_1)$ and $\nu(\theta_1+\phi)$ as shown in Fig. 3,

$$\begin{aligned} E &= \frac{\langle E,\ \nu(\theta_1+\phi)^\perp\rangle}{\langle \tau(\theta_1),\ \nu(\theta_1+\phi)^\perp\rangle}\tau(\theta_1) + \frac{\langle E,\ \tau(\theta_1)^\perp\rangle}{\langle \nu(\theta_1+\phi),\ \tau(\theta_1)^\perp\rangle} \\ &\quad \times \nu(\theta_1+\phi) \\ &= \frac{\langle E,\ \tau(\theta_1+\phi)\rangle}{\cos\phi}\,\tau(\theta_1) + \frac{\langle E,\ \nu(\theta_1)\rangle}{\cos\phi}\,\nu(\theta_1+\phi). \end{aligned}$$

The stabilizing controller adds correcting terms proportional to the error components to Eq. (8). To ensure that the physical constraint of the follower is satisfied, the correcting terms are multiplied by a time-varying gain $\eta(t)$ that must be chosen in a suitable way (see Eq. (29)). In the following theorem $\mathcal{S}^1$ denotes the quotient space $\mathbb{R}/\mathfrak{R}$ equipped with the canonical topology being $\mathfrak{R}$ the equivalence relation, $x \sim y \Leftrightarrow x - y = 2n\pi$, $n \in \mathbb{Z}$, where $\mathbb{Z}$ is the integer set.

Theorem 2. *Set $d > 0$, $\phi : |\phi| < \frac{\pi}{2}$ and $\overline{\mathbf{R}}_0, \overline{\mathbf{R}}_1 \in \mathbb{R}^3$. Let $\mathbf{R}_0 = (P_0^{\mathrm{T}}, \theta_0)$ be a robot with initial condition $\overline{\mathbf{R}}_0$, satisfying the trajectory constraint (V_0, K_0^-, K_0^+). Suppose that the following properties hold:*

$$0 < W_0 \le v_0(t) \tag{18}$$

$$\begin{aligned} &-\frac{1}{d} < K_0^- \le K_0^+ < \frac{1}{d\cos\phi}, \quad \textit{if } \phi \ge 0 \\ &-\frac{1}{d\cos\phi} < K_0^- \le K_0^+ < \frac{1}{d}, \quad \textit{if } \phi < 0 \end{aligned} \tag{19}$$

$$\widetilde{K}_0^- < K_0^- \le K_0^+ < \widetilde{K}_0^+ \tag{20}$$

$$\begin{aligned} &V_0 \cos(0 \wedge (\arcsin(K_0^+ d\cos\phi) - \phi) \\ &\quad \wedge (\phi - \arcsin(K_0^- d\cos\phi))) < V_p\ \cos\phi \end{aligned} \tag{21}$$

where

$$\widetilde{K}_0^\pm = (\ \textit{sign}\ K_p^\pm)(((K_p^\pm)^{-1} - d\sin\phi)^2 + d^2\cos^2\phi)^{-\frac{1}{2}}. \tag{22}$$

Then there exists $\bar\epsilon > 0$ such that for any $\epsilon : 0 < \epsilon < \bar\epsilon$, for any robot $\mathbf{R}_1 = (P_1^{\mathrm{T}}, \theta_1)^{\mathrm{T}}$ with initial condition $\overline{\mathbf{R}}_1$, there exist suitable controls v_1, ω_1 such that $\mathbf{R}_0$ and $\mathbf{R}_1$ are asymptotically in (d, ϕ)-formation and $\exists\, \bar t \ge 0 : \forall\, t \ge \bar t$

$$\begin{aligned} &\arcsin((K_0^- - \epsilon)\, d\cos\phi) \le \theta_0(t) - \theta_1(t) \\ &\quad \le \arcsin((K_0^+ + \epsilon)\, d\cos\phi). \end{aligned} \tag{23}$$

Proof. Take $\bar\epsilon > 0$ such that

$$\begin{aligned} &-\frac{1}{d} < K_0^- - \bar\epsilon \le K_0^+ + \bar\epsilon < \frac{1}{d\cos\phi}, \quad \text{if } \phi \ge 0 \\ &-\frac{1}{d\cos\phi} < K_0^- - \bar\epsilon \le K_0^+ + \bar\epsilon < \frac{1}{d}, \quad \text{if } \phi < 0 \end{aligned} \tag{24}$$

$$\widetilde{K}_0^- < K_0^- - \bar\epsilon \le K_0^+ + \bar\epsilon < \widetilde{K}_0^+ \tag{25}$$

$$\begin{aligned} &V_0 \cos(0 \wedge (\arcsin((K_0^+ + \bar\epsilon)\, d\cos\phi) - \phi) \\ &\quad \wedge (\phi - \arcsin((K_0^- - \bar\epsilon)\, d\cos\phi))) < V_p\ \cos\phi \end{aligned} \tag{26}$$

and take any $\epsilon : 0 < \epsilon \le \bar\epsilon$. Define $\Gamma_\epsilon = \{x \in \mathcal{S}^1 | (K_0^- - \epsilon)\, d\cos\phi \le \sin x \le (K_0^+ + \epsilon)\, d\cos\phi\}$. Set the controls as

$$v_1(t) = \begin{cases} V_p, & \text{if } \theta_0(t) - \theta_1(t) \notin \Gamma_\epsilon \\ \dfrac{v_0(t)\cos(\theta_0(t) - \theta_1(t) - \phi) + \eta(t)\langle E(t),\ \tau(\theta_1(t)+\phi)\rangle}{\cos\phi}, \\ \quad \text{if } \theta_0(t) - \theta_1(t) \in \Gamma_\epsilon \end{cases} \tag{27}$$

$$\omega_1(t) = \begin{cases} V_p\ K_p^+, & \text{if } \theta_0(t) - \theta_1(t) \notin \Gamma_\epsilon \text{ and } K_p^+ \ge 0 \\ V_p\ K_p^-, & \text{if } \theta_0(t) - \theta_1(t) \notin \Gamma_\epsilon \text{ and } K_p^+ < 0 \\ \dfrac{v_0(t)\sin(\theta_0(t) - \theta_1(t)) + \eta(t)\ \langle E(t),\ \nu(\theta_1(t))\rangle}{d\cos\phi}, \\ \quad \text{if } \theta_0(t) - \theta_1(t) \in \Gamma_\epsilon \end{cases} \tag{28}$$

where $\eta(t)$ is a function given by

$$\begin{aligned} \eta(t) = {} & \frac{(v_0 - W_0/2)\cos(\theta_0 - \theta_1 - \phi)}{|\langle E,\ \tau(\theta_1+\phi)\rangle|} \\ & \wedge \frac{[(K_0^+ + \epsilon/2 - \kappa_0) \wedge (\kappa_0 - (K_0^- - \epsilon/2))]\, d\cos\phi}{|\langle E,\ \nu(\theta_1)\rangle|} \\ & \wedge \frac{V_p\cos\phi - v_0\cos(\theta_0 - \theta_1 - \phi)}{|\langle E,\ \tau(\theta_1+\phi)\rangle|} \\ & \wedge v_0 \frac{K_p^+ d\cos(\theta_0 - \theta_1 - \phi) - \sin(\theta_0 - \theta_1)}{|\langle E,\ \nu(\theta_1)\rangle| + |K_p^+||\langle E,\ \tau(\theta_1+\phi)\rangle|} \\ & \wedge v_0 \frac{\sin(\theta_0 - \theta_1) - K_p^- d\cos(\theta_0 - \theta_1 - \phi)}{|\langle E,\ \nu(\theta_1)\rangle| + |K_p^-||\langle E,\ \tau(\theta_1+\phi)\rangle|} \wedge M \end{aligned} \tag{29}$$

and M is a positive gain constant (with the convention that $\frac{1}{0} = +\infty$). Set $\beta(t) = \theta_0(t) - \theta_1(t)$, $\forall\ t \ge 0$. First of all we remark that $\exists\ \bar t \ge 0 : \beta(\bar t) \in \Gamma_\epsilon$. In fact, suppose for instance that $K_p^+ \ge 0$, then if $\forall\ t \ge 0$, $\beta(t) \notin \Gamma_\epsilon$, by (25): $\dot\beta(t) = \omega_0(t) - \omega_1(t) = \kappa_0(t)\ v_0(t) - K_p^+ V_p \le$

$K_0^+ v_0(t) - K_p^+ V_p \le (\widetilde{K}_0^+ v_0(t) - K_p^+ V_p) - \epsilon v_0(t) \le (\widetilde{K}_0^+ V_0 - K_p^+ V_p) - \epsilon W_0 \le -\epsilon W_0$, being $\widetilde{K}_0^+ V_0 - K_p^+ V_p \le 0$. In fact, suppose for simplicity that $\cos(0 \wedge (\arcsin((K_0^+ + \epsilon) d \cos\phi) - \phi) \wedge (\phi - \arcsin((K_0^- - \epsilon) d \cos\phi))) = \cos(0) = 1$, by (11) and (26) $\widetilde{K}_0^+ V_0 - K_p^+ V_p \le \{((K_p^+)^{-1} - d \sin\phi)^2 + d^2 \cos^2\phi\}^{-1/2} V_p \cos\phi - K_p^+ V_p \le V_p \{((K_p^+)^{-1} - d\sin\phi)^2 + d^2\cos^2\phi\}^{-1/2} (\cos\phi - \{\cos^2\phi + (K_p^+ d - \sin\phi)^2\}^{1/2}) \le 0$. Therefore $\beta(t) \le -\epsilon W_0 t + \beta(t_0)$, $\forall\, t \ge 0$, which implies straightaway property (23). Set $\beta^{-1}(\Gamma_\epsilon) = \{t \ge 0 | \beta(t) \in \Gamma_\epsilon\}$, then by (28), $\forall\, t \in \beta^{-1}(\Gamma_\epsilon)$: $\dot\beta(t) = \omega_0(t) - \omega_1(t) = \omega_0(t) - \frac{v_0(t)\sin\beta(t) + \eta(t)\langle E(t), \nu(\theta_1(t))\rangle}{d\cos\phi} = \frac{v_0(t)}{d\cos\phi}(\kappa_0(t)\, d\cos\phi - \sin\beta(t)) - \frac{\eta(t)\langle E(t), \nu(\theta_1(t))\rangle}{d\cos\phi}$, which implies by definition (29) of $\eta(t)$ that if $\beta(\bar t) = \arcsin((K_0^+ + \epsilon) d\cos\phi)$, $(\beta(\bar t) = \arcsin((K_0^- - \epsilon)\, d\cos\phi))$ then $\dot\beta(\bar t) \le -\frac{\epsilon V_0}{2}$, $(\dot\beta(\bar t) \ge \frac{\epsilon W_0}{2})$. This implies that $\forall\, t \in [0, \bar t)$

$$\exists\, \bar t \ge 0 \text{ such that } \forall\, t \ge \bar t,\, \beta(t) \in \Gamma_\epsilon \text{ and } \beta(t) \notin \Gamma_\epsilon \tag{30}$$

which implies (23). To prove that the physical constraint (11) is verified for v_1, remark that by (27) and (30), $v_1(t) = V_p$, $\forall\, t \in [0, \bar t)$ and $\forall\, t \ge \bar t$

$$v_1(t) = \frac{v_0(t)\cos(\beta(t) - \phi) + \eta(t)\langle E(t), \tau(\theta_1(t) + \phi)\rangle}{\cos\phi}. \tag{31}$$

Therefore by definition of $\eta(t)$ and (18), $v_1(t) \ge \frac{v_0(t)\cos(\beta(t)-\phi) - \eta(t)\langle E(t), \tau(\theta_1(t)+\phi)\rangle}{\cos\phi} \ge \frac{W_0}{2}\cos(\beta(t) - \phi)$. Therefore $v_1(t) > 0, \forall\, t \ge 0$, by the following property:

$$\exists\, c_1 > 0 : \cos(\beta(t) - \phi) \ge c_1,\ \forall\, t \ge \bar t. \tag{32}$$

In fact suppose for instance that $\phi > 0$, by (23) and (24) $\beta(t) - \phi \le \arcsin((K_0^+ + \epsilon)\, d\cos\phi) - \phi < \arcsin(1) - \phi = \frac{\pi}{2} - \phi$ and $\beta(t) - \phi \ge \arcsin((K_0^- - \epsilon)\, d\cos\phi) - \phi > \arcsin(\cos\phi) - \phi = -\frac{\pi}{2}$ which implies (32). Furthermore by (31) and (29), $v_1(t) \le V_p, \forall\, t \ge 0$. To verify constraint (11) for $\kappa_1(t)$, remark first of all that it is verified by (27) and (28) if $t \in [0, \bar t)$. If $t \ge \bar t$: $\kappa_1(t) = \frac{\omega_1(t)}{v_1(t)} = \frac{v_0(t)\sin\beta(t) + \eta(t)\langle E(t), \nu(\theta_1(t))\rangle}{d v_0(t)\cos(\beta(t)-\phi) + \eta(t)\langle E(t), \tau(\theta_1(t)+\phi)\rangle}$ $\kappa_1(t) \le \frac{v_0(t)\sin\beta(t) + \eta(t)|\langle E(t), \nu(\theta_1(t))\rangle|}{d v_0(t)\cos(\beta(t)-\phi) - \eta(t)\ \mathrm{sign}\ K_p^+ |\langle E(t), \tau(\theta_1(t)+\phi)\rangle|} \le K_p^+$, since, by (29), $\eta(t) \le \frac{v_0(t)(K_p^+ d\cos(\beta(t)-\phi) - \sin\beta(t))}{|\langle E(t), \nu(\theta_1(t))\rangle| + |K_p^+||\langle E(t), \tau(\theta_1(t)+\phi)\rangle|}$ and analogously, $\kappa_1(t) \ge K_p^-$, $\forall\, t \ge \bar t$; therefore constraint (11) is completely verified. To conclude the proof it remains to verify that $\mathbf{R}_0$ and $\mathbf{R}_1$ are asymptotically in (d, ϕ)-formation. Differentiating the error $E(t)$, by (27) and (28), as in the proof of Theorem 1, we get that $\forall\, t \ge \bar t$: $\dot E = v_0(t)\ \tau(\theta_0(t)) - v_0(t)(\frac{\cos(\beta(t)-\phi)}{\cos\phi}\ \tau(\theta_1(t)) + d\frac{\sin\beta(t)}{\cos\phi}\ \nu(\theta_1(t) + \phi)) - \eta(t)(\frac{\langle E(t), \tau(\theta_1(t)+\phi)\rangle}{\cos\phi}\ \tau(\theta_1(t)) + \frac{\langle E(t), \nu(\theta_1)\rangle}{\cos\phi}\ \nu(\theta_1(t) + \phi)) = -\eta\ E(t)$, since $\forall\ w, w_1, w_2 \in \mathbb{R}^2$ such that $\langle w_1, w_2^\perp\rangle \ne 0$, $w = \frac{\langle w, w_2^\perp\rangle}{\langle w_1, w_2^\perp\rangle}\ w_1 + \frac{\langle w, w_1^\perp\rangle}{\langle w_2, w_1^\perp\rangle}\ w_2$. Therefore $\frac{d}{dt}(\|E(t)\|^2) = -2\eta(t)(\|E(t)\|^2)$, $\forall\, t \ge \bar t$ and then $\lim_{t\to+\infty} E(t) = 0$ since the following property holds,

$$\exists\, c > 0 \text{ such that } \eta(t) \ge \left(\frac{c}{\|E(t)\|} \wedge M\right). \tag{33}$$

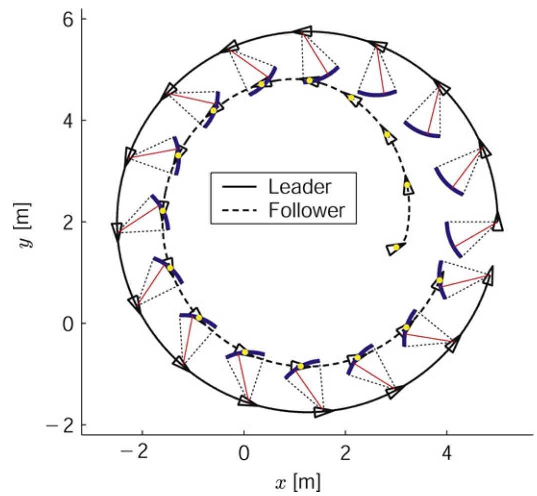

Fig. 4. Trajectory of the robots and arcs of circle.

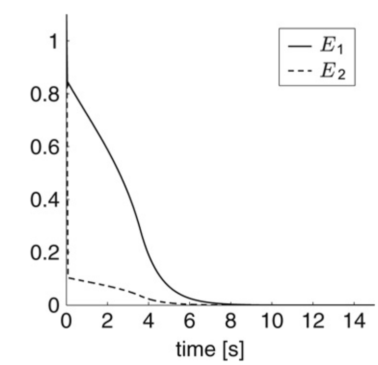

Fig. 5. Error vector $E = (E_1, E_2)^{\mathrm{T}}$.

In fact to verify (33), first of all remark that by (18) it follows straightaway that $\left(K_0^+ + \frac{\epsilon}{2} - \kappa_0(t)\right) \wedge \left(\kappa_0(t) - (K_0^- - \frac{\epsilon}{2})\right) \ge \frac{\epsilon}{2}$, $\forall\, t \ge 0$. Moreover by (26), $V_p\cos\phi - v_0(t)\cos(\theta_0(t) - \theta_1(t) - \phi) \ge V_p\cos\phi - V_0\cos(0 \wedge (\arcsin(K_0^+ d\cos\phi) - \phi) \wedge (\phi - \arcsin(K_0^- d\cos\phi))) = c_2 > 0$ and furthermore by (23), $\forall\, t \ge \bar t$, $K_p^+ - \frac{\sin\beta(t)}{d\cos(\beta(t)-\phi)} \ge K_p^+ - \frac{\mathrm{sign}\ (K_0^+ + \epsilon)}{\sqrt{(K_0^+ + \epsilon)^{-2} - d^2\cos^2\phi} + d\sin\phi} > 0$, since $(K_0^+ + \epsilon) < \widetilde{K}_0^+$, which implies that $\exists\, c_3^+ > 0$ such that $K_p^+ d\cos(\beta(t) - \phi) - \sin\beta(t) \ge c_3^+$; analogously $\exists\, c_3^- > 0$ such that $\sin\beta(t) - K_p^- d\cos(\beta(t) - \phi) \ge c_3^-$. Therefore bringing together (32) with the previous inequalities, we obtain (33) by the definition of η. □

5. Simulation results

Figs. 4 and 5 show the results of the simulation experiment we conducted to test the effectiveness of the stabilizing controller presented in Section 4. The leader $\mathbf{R}_0$ moves along a circular path with $v_0(t) = 1.5$ m/s, $\omega_0(t) = 0.4$ rad/s. The initial conditions are $\overline{\mathbf{R}}_0 = (5, 2, \pi/2)$, $\overline{\mathbf{R}}_1 = (3, 1.5, \pi/8)$ and $d = 1$ m, $\phi = -\pi/3$ rad. Moreover, $W_0 = 1$ m/s, $V_0 = 2$ m/s, $K_0^- = -1$ rad/m, $K_0^+ = 0.5$ rad/m, $V_p = 4$ m/s, $K_p^- = -2$ rad/m, $K_p^+ = 3$ rad/m, $\epsilon = 0.05$ rad/m and $M = 1$ rad/m. Fig. 4 depicts the trajectory of the leader $\mathbf{R}_0$ (solid), the follower $\mathbf{R}_1$ (dash) and the arcs of circle $\mathcal{A}_d(\theta_0(t) + \phi + \pi, -\arcsin((K_0^+ + \epsilon)\ d\ \cos\phi), -\arcsin((K_0^- - \epsilon)\ d\ \cos\phi))$ introduced in Remark 1 (the position of the robots is reported in Fig. 4 each second). $\mathbf{R}_0$ and $\mathbf{R}_1$ are in $(1, -\pi/3)$-formation approximately at time $t = 6$ s. At steady state $\mathbf{R}_1$ keeps on the arcs of circle. Since the bounds and parameters chosen in the simulation experiment satisfy Eqs. (18) and (24)–(26), by applying controls (27), (28) from Theorem 2 $\lim_{t\to+\infty} E(t) = 0$ (see Fig. 5), and conditions (11) and (23) hold.

6. Conclusions

The paper introduces an original leader–follower formation control strategy for unicycle robots with input constraints. Precise conditions for formation keeping and asymptotic stabilization are proved.

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