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Introducing the "SPHERICLE": an Experimental Testbed for Research and Teaching in Nonholonomy*

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Abstract

In this paper we describe an experimental apparatus developed in our laboratory for research and advanced teaching purposes. The device consists in an untethered spherical vehicle that autonomously rolls on the laboratory floor, and can reach arbitrary positions and orientations in the environment. The kinematics of the vehicle are nonholonomic, and result from the combination of the kinematics of two classical nonholonomic systems, namely, a unicycle and a plate–ball system. The "sphericle" however introduces features that are new with respect to both, which fact renders its study particularly interesting.

1 Introduction

Nonholonomy in the kinematics and dynamics of mechanical systems has been attracting much attention in the robotics and control communities in recent years. Nonholonomic systems arise very often in practice, as for instance in car–like vehicles, tractor–trailer systems, airborne and underwater vehicles. Nonholonomic behaviours are sometimes introduced on purpose in the design of mechanism, in order to obtain certain characteristics and performances: such is the case for instance with work described in [21], [23], [1], and [6].

One advantage offered by nonholonomic systems is the possibility of controlling a higher number of configurations than the number of actuators actually employed in the system, which fact is sometimes useful in terms of reducing the system's weight and cost, and increases its reliability. Full exploitation of nonholonomic systems is however hindered by the intrinsic difficulties of planning and controlling them. Nonholonomic systems are infact one of the most significant classes of *intrinsically nonlinear* systems, i.e., systems to which approximated linearization methods can not be applied, without destroying some of the structural properties of the system itself. The wide attention that the study of nonholonomic systems has received has produced a wealth of methods that apply to different classes of systems. Although many open problems remain (which fact makes the study of nonholonomy one of the most challenging to roboticists and control theorists alike), an issue of transferring knowledge about such new results and methods to researchers fresh in the area, and to graduate students, is also in order.

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2 System Description

The spherical vehicle (nicknamed "The Sphericle") built in our laboratory consists of a hollow ball rolling freely on the floor ([13]). The outer surface of the robot is a perfectly smooth sphere, painted in black with marks to render the orientation of the sphere evident. The vehicle is powered autonomously; its logic is partly implemented on–board, and partly in a base station, connected through a radio modem. No tether is used.

A typical demonstration task of the sphericle is to move in a complex environment such as a house from a room to another, place itself in the correct posture, and convey a vocal message. Several useful tasks can be conceived for the sphericle e.g. in inspection and surveillance, although practical applications of the vehicle are not an issue in this paper.

To make the sphericle move, a mobile mass is placed within the cavity of the ball. This can be realized in different ways. For instance, Koshiyama and Yamafuji [12] built an omnidirectional steering robot with a spherical wheel, an arch–shaped body mounted over the ball, and an arm–like mechanism inside the ball. Halme *et al.* [9] built a spherical rolling robot that uses a wheeled device enclosed in the sphere cavity to actuate the motion. The latter design is similar to the one described in [13] and in this paper.

In our implementation, the moving mass is comprised of a small car with unicycle kinematics, its actuators, drivers, sensors, battery pack, and radio modem (see fig.2). The car is inserted in the ball through an opening on the sphere, which is sealed afterwards to recover a perfect spherical shape. The car is kept by its own weight in contact with the sphere, onto the inner surface of which the car wheels roll without slip-

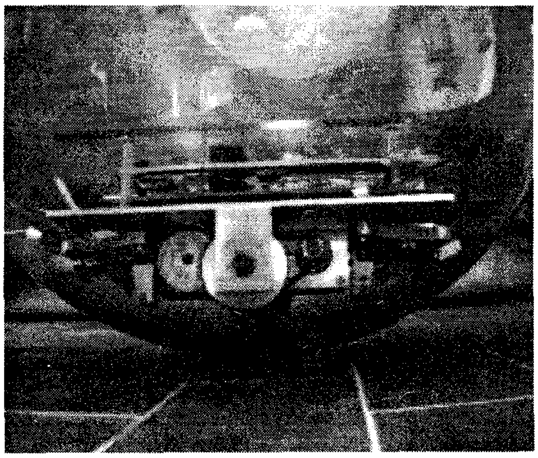

Figure 1: The prototype in our laboratory

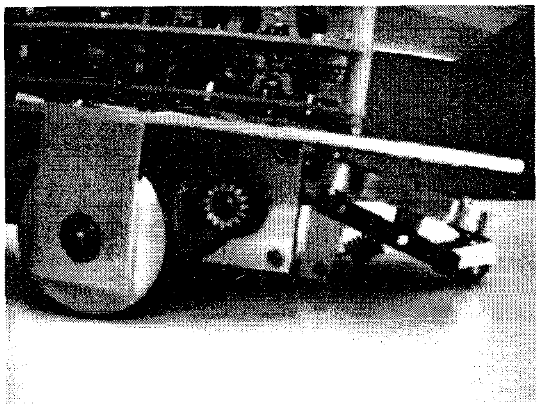

Figure 2: A close-up of the inner vehicle showing the stepper motors and the suspension system

ping. To make the system more robust with respect to external perturbations, a boom can be mounted on the car so as to maintain contact on the sphere's ceiling, thus forcing the car onto the sphere floor more effectively ([9]). In our prototype, elastic suspensions are mounted on the front and back of the unicycle (see fig.2). Two stepper motors with 200 steps per revolution are used, with a 3.2 belt reduction gear. The stepper circuit is driven by a square wave generated directly by a microcontroller TI TMS370C756 which is mounted onboard. The μc receives high-level planning instructions from an off-board computer via a two-way serial radio link at 19200 bps (manufactured by Astrel, mod.297), guaranteeing communications to within ca. 80 m. in a noisy environment. A free pendulum is mounted on the cart, its angle being measured by an encoder. The μc uses the wheels and pendulum positions to implement a local feedback stabilization controller, which is superimposed to the path following commands, which are currently realized in open loop.

3 Modeling

In this section we report on the mathematical models that are used to plan and control the motions of the Sphericle. We describe first a quasi-static model of the system, which ignores its dynamics, but nonetheless captures many interesting aspects of its nonholonomy. In the second part, we consider the dynamics of the system.

3.1 Kinematics

As already mentioned, the kinematics of the Sphericle are a combination of those of a unicycle and of a plate-ball system. The kinematics of these systems are recalled for the reader's convenience.

3.1.1 Unicycle kinematics

For a unicycle moving in a x–y plane, whose orientation is described by the angle θ, the kinematic equations of motion are

$$\begin{cases} \dot{x} = C_\theta \, v \\ \dot{y} = S_\theta \, v \\ \dot{\theta} = w \end{cases} .$$

where v is the forward velocity of the vehicle (i.e., the average of the angular velocities of the wheels times their radius), w is the angular velocity of the vehicle (corresponding to the difference of the wheel velocities times the ratio of their radius to half their axial distance), and S_θ, C_θ indicate the sine and cosine of θ. This system, considered as a nonlinear control system linear in the controls (v and w), is an almost ubiquitous example of nonholonomy, about which some well known facts are:

1. it is controllable, i.e., for any given initial and final configurations, there exists at least one control $v(t), w(t)$, that joins them. The system is intrinsically nonlinear: in fact, the approximate linearization of the model destroys its controllability. To show controllability, one level of Lie bracketing is sufficient, hence the system's degree of nonholonomy is one;
2. it is not linearizable by static state feedback ([10]);
3. it is not stabilizable to an equilibrium point by a smooth, time-invariant state feedback ([3]). However, it is smoothly stabilizable to a two-dimensional submanifold of the configuration space, e.g., to track a path in the plane ([5]);
4. it is feedback linearizable by dynamic extension ([5]), or, equivalently, is differentially flat ([22]). Moreover, the system can be put in one-chained form ([19]) and hence is nilpotentizable [14].

3.1.2 Plate-Ball Kinematics

The kinematics of a sphere rolling on the plane are conveniently described in terms of a local parametrization of the configuration manifold $\mathbb{R}^2 \times SO(3)$ proposed by Montana [17]. This consists in taking as coordinates the position x, y of the contact point in a orthogonal reference frame fixed to the plate, the azimuth u and elevation v of the contact point in a spherical coordinate reference frame fixed to the ball, and the holonomy angle ψ between the x–axes of the plate and ball Gauss frames at the contact point. Such description is not globally valid, as the north and south poles of the sphere are singularity points for the chosen coordinates. A suitable change of coordinates should be applied when in a neighborhood of singularities.

In the assumption that the ball rolls without slipping on the plate, and that its rotations about an axis normal to the plate through the contact point ("spinning") is prevented by friction, the equations of motion are as follows

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{u} \\ \dot{v} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 \\ -R \\ \frac{S_\psi}{C_v} \\ C_\psi \\ T_v S_\psi \end{bmatrix} w_x + \begin{bmatrix} R \\ 0 \\ \frac{C_\psi}{C_v} \\ -S_\psi \\ T_v C_\psi \end{bmatrix} w_y,$$

or, compactly, as

$$\dot{\mathbf{x}} = \mathbf{g}_1(\mathbf{x}) w_x + \mathbf{g}_2(\mathbf{x}) w_y \qquad (1)$$

where the controls w_x, w_y are the components of the angular velocity of the ball with respect to the plate, which appear linearly in the nonlinear system equations. The plate–ball kinematics form a two–inputs, five–states system, which also received some attention in the literature. Some known results are as follows:

1. it is controllable, and intrinsically nonlinear. The degree of nonholonomy is three ([15]);
2. it is not linearizable by static state feedback, nor smoothly stabilizable to an equilibrium point;
3. it is not differentially flat ([22]), it is not nilpotentizable ([8]), it can not be put in chained form ([20]), and it cannot be exactly discretized [16].
4. it can be put in strictly triangular form by smooth state feedback and change of coordinates ([2]);

To verify the above properties is an interesting exercise. For instance, the fact that the plate–ball system cannot be put in chained form follows from the discussion in ([19]), regarding the relative growth vector of the controllability filtration of a chained system. Letting $[\mathbf{f}, \mathbf{g}]$ denote the Lie bracket between two vector fields,

$$[\mathbf{f}, \mathbf{g}] = \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \mathbf{g},$$

the *controllability filtration* of the nonlinear control system (1) is defined as the *iterative* sequence of distributions ([10])

$$\begin{array}{rcl} \Delta_0 & = & \Delta, \\ \Delta_1 & = & \Delta_0 + [\Delta_0, \Delta_0], \\ \Delta_2 & = & \Delta_1 + [\Delta_1, \Delta_0], \\ & \vdots & \\ \Delta_{i+1} & = & \Delta_i + [\Delta_i, \Delta_0]. \end{array} \qquad (2)$$

where $\Delta = \mathsf{span}\ \{\mathbf{g}_1(\mathbf{x}), \mathbf{g}_2(\mathbf{x})\}$ is the input distribution. The above mentioned necessary condition for conversion to chained form is that $\mathsf{dim}\ \Delta_i = i + 2$. By direct computation of the Lie brackets of the control vector fields, i.e. $[\mathbf{g}_1, \mathbf{g}_2]$, $[\mathbf{g}_1, [\mathbf{g}_1, \mathbf{g}_2]]$, and $[\mathbf{g}_2, [\mathbf{g}_1, \mathbf{g}_2]]$, and by checking their linear independence, it is easily verified that the condition is not met, as $\mathsf{dim}\ \Delta_0 = 2$, $\mathsf{dim}\ \Delta_1 = 3$, but $\mathsf{dim}\ \Delta_2 = 5$. This count also confirms controllability of the plate–ball system.

We remark two interesting facts about the plate–ball kinematics:

Remark 1. Any system with 4 states and at least two inputs can be put in chained form, hence is nilpotent and differentially flat. Planning and controlling such systems can be considered a solved problem, at least thoeoretically (see e.g. [16], [14], [19]). Thus, the plate–ball system represents one of the simplest possible systems to which no known systematic planning/control algorithm applies;

Remark 2. Brockett and Dai [4] and Jurdjevic [11] investigated the problem of shortest paths for a ball rolling on a plate. Both authors put in evidence how very classical mathematical problems are intimately connected with this one, the former paper discussing the role of elliptical functions in solving it, and the latter showing the equivalence with the famous *elastica* problem of Euler.

3.1.3 Sphericle Kinematics

In the operation of the robot for very slow speeds, a quasi–static kinematic model of the sphericle can be obtained by linking the unicycle and plate–ball systems through the constraints that

- the vertical projection of the center of mass of the unicycle on the ground coincides with the contact point between the sphere and the floor;
- friction between the sphere and the floor is large, so that the sphere rolls without slipping nor spinning.

The sphericle's kinematics are therefore described by an equation

$$\dot{\xi} = \mathbf{g}_1(\xi) v + \mathbf{g}_2(\xi) w, \qquad (3)$$

where $\xi = [\ \dot{x} \quad \dot{y} \quad \dot{u} \quad \dot{v} \quad \dot{\psi} \quad \dot{\theta}\ ]^T$,

$$\mathbf{g}_1(\xi) = \left[C_\theta \ S_\theta \ \frac{C_{\theta+\psi}}{RC_v} \ -\frac{S_{\theta+\psi}}{R} \ \frac{T_v C_{\theta+\psi}}{R} \ 0 \right],$$

and

$$\mathbf{g}_2(\xi) = [\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1\ ] \omega.$$

The new system has two inputs and six states. If we let $\mathbf{g}_1$, $\mathbf{g}_2$ denote now the two control vector fields in (3), then easy computations show that $\mathsf{dim}\ \Delta_1 = \mathsf{dim}\ (\Delta_0 + \mathsf{span}\ \{[\mathbf{g}_1, \mathbf{g}_2]\}) = 3$; $\mathsf{dim}\ \Delta_2 = \mathsf{dim}\ (\Delta_1 + \mathsf{span}\ \{[\mathbf{g}_1, [\mathbf{g}_1, \mathbf{g}_2]]\}) = 4$; $\mathsf{dim}\ \Delta_3 = \mathsf{dim}\ (\Delta_2 + \mathsf{span}\ \{[\mathbf{g}_1, [\mathbf{g}_1, [\mathbf{g}_1, \mathbf{g}_2]]]\}) = 5$, and finally $\mathsf{dim}\ \Delta_4 = \mathsf{dim}\ (\Delta_3 + \mathsf{span}\ \{[\mathbf{g}_2, [\mathbf{g}_1, [\mathbf{g}_1, [\mathbf{g}_1, \mathbf{g}_3]]]]\}) = 6$.

From the above calculations, it turns out that the sphericle is completely controllable to an arbitrary 6-dimensional configuration (position and orientation of the sphere, direction of the unicycle within) by using the unicycle wheels as inputs. Note that we did not address the existence of singularities in the chosen coordinate set, i.e. at $v = \pm\frac{\pi}{2} + k\pi,\ k = \pm 1, \pm 2, \ldots$. Related problems can be easily circumvented by repeating similar calculations in a different coordinate set.

Moreover, the necessary condition of [19] for conversion to chained form is satisfied by the kinematic equations of the sphericle. However, application of the recent results of Murray [18] show that the system can not actually be put in chained form. According to such theorem, in fact, a necessary and sufficient condition for a system to be convertible to chained form is that a) it satisfies the above condition of on the growth of the controllability filtration, and b) it satisfies the same condition on the corresponding ***Goursat filtration***. The latter is defined as the ***recursive*** sequence of distributions

$$\Gamma_0 = \Delta, \tag{4}$$
$$\Gamma_1 = \Gamma_0 + [\Gamma_0, \Gamma_0], \tag{5}$$
$$\Gamma_2 = \Gamma_1 + [\Gamma_1, \Gamma_1], \quad \vdots \quad \Gamma_{i+1} = \Gamma_i + [\Gamma_i, \Gamma_i]. \tag{6}$$

By computing the Goursat filtration, it is found that $\Gamma_1 = \Delta_1$ and $\Gamma_2 = \Delta_2$, while for the sphericle $\Gamma_3 = \Delta_3 + \text{span}\,\{[[g_1, g_2], [g_1, [g_1, g_2]]]\}$, and $\dim \Gamma_3 = 6$.

Remark 3. Because for any 5 dimensional nonsingular system the dimension of the controllability and Goursat filtrations always coincide, the sphericle can be considered as one of the simplest devices for which the two conditions are different. Our example is dual to the one reported by Giaro *et al.* ([7]), where only the condition on the Goursat filtration is met.

The sphericle can not be put in strictly triangular form by the transformation used for general objects rolling on a plane ([2]). However, a (not strictly) triangular form can be achieved by the following state feedback:

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} -\frac{R}{S_{\theta+\psi}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}$$

Indeed we get

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{u} \\ \dot{v} \\ \dot{\psi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -\frac{RC_\theta}{S_{\theta+\psi}} \\ -\frac{RS_\theta}{S_{\theta+\psi}} \\ -\frac{1}{C_v T_{\theta+\psi}} \\ 1 \\ -\frac{T_v}{T_{\theta+\psi}} \\ 0 \end{bmatrix} \hat{u}_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \hat{u}_2 \tag{7}$$

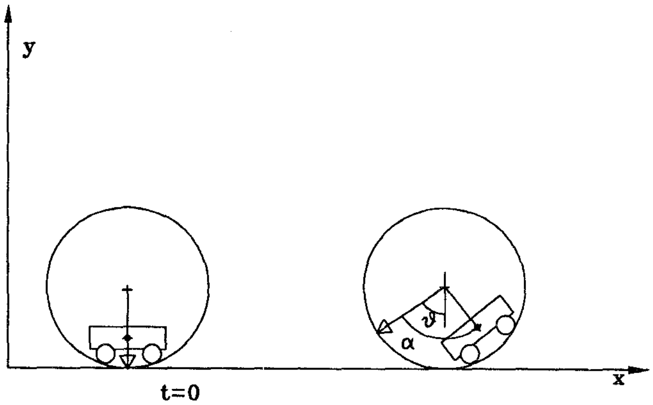

Figure 3: Dynamic model of the sphericle moving in a plane

3.2 Dynamics

Actual implementation of the sphericle experiment shows how the quasi-static kinematic model above discussed is only valid in a very narrow range of operating conditions. The system has an intrinsic tendency to oscillate, which makes it rather difficult to control in open-loop along planned trajectories. Moreover, even slight irregularities in the terrain cause perturbations that are very slowly eliminated without active stabilization of the system.

In order to allow design and implementation of an effective stabilization of the sphericle, a dynamic model of the system is necessary. In this paper, we confine ourselves to a planar model, described in fig.3.2. For simplicity sake, the unicycle moving inside the sphericle is modelled as a point mass m, and the sphere is assumed to have mass M, radius R, and moment of inertia I. As Lagrangian coordinates for the system we choose the angle θ between the radius through a fixed point on the sphere and the vertical directon, and the angle α between the same radius and the radius through the unicycle center (see fig.3.2). The kinetic energy of the sphere E_s and of the unicycle E_u, and the gravitational potential of the unicycle V_u, are evaluated as

$$\begin{aligned} E_s &= \frac{1}{2}(MR^2 + I)\dot{\theta}^2 \\ E_c &= \frac{1}{2}mR^2\left[\dot{\theta}^2 + (\dot{\alpha} - \dot{\theta})^2 + 2\dot{\theta}C_{\alpha-\theta}(\dot{\alpha} - \dot{\theta})\right] \\ V_u &= mgR\left[1 - C_{\alpha-\theta}\right] \end{aligned}$$

By applying the Lagrangian equation of motion,

$$\frac{d}{dt}\left(\frac{\partial E_s + E_u}{\partial \dot{q}_r}\right) - \frac{\partial E_s + E_u - V_u}{\partial q_r} = \tau_r, \qquad r = 1, 2$$

where $q_1 = \theta$, $q_2 = \alpha$, and τ_r are the nonconservative generalized forces corresponding to the chosen coordinates, after some calculations we get

$$\mathbf{M}_{(\mathbf{q})}\ddot{\mathbf{q}} + \mathbf{C}_{(\mathbf{q},\dot{\mathbf{q}})}\dot{\mathbf{q}} + \mathbf{h}(\mathbf{q}) = \tau$$

where

$$\mathbf{M}_{(\mathbf{q})} = \begin{bmatrix} MR^2 + I + 2mR^2(1 - C_{\alpha-\theta}) & -mR^2(1 - C_{\alpha-\theta}) \\ -mR^2(1 - C_{\alpha-\theta}) & mR^2 \end{bmatrix};$$

$$\mathbf{C}_{(\mathbf{q},\dot{\mathbf{q}})} = \begin{bmatrix} mR^2 S_{\alpha-\theta}(\dot\alpha - \dot\theta) & -mR^2 S_{\alpha-\theta}(\dot\alpha - \dot\theta) \\ 0 & 0 \end{bmatrix};$$

$$\mathbf{h}(\mathbf{q}) = \begin{bmatrix} -mgRS_{\alpha-\theta} \\ mgRS_{\alpha-\theta} \end{bmatrix},$$

and $\tau = [\tau_\theta \ \tau_\alpha]^T$. For actuation at the torque level on the unicycle wheels (assumed of radius r), and in unperturbed conditions, it is in particular $\tau = [0 \ \frac{R}{r}\tau_{motor}]^T$. In this case, the dynamics can be written in state–space form as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\tau_{motor}$, where $\mathbf{x} = [\theta \ \alpha \ \dot\theta \ \dot\alpha]^T$, and

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \dot\theta \\ \dot\alpha \\ \frac{-2\,m\,R\left(\dot\alpha^2 R - 2\,\dot\alpha\,\dot\theta\,R + \dot\theta^2 R + g\,C_{\alpha-\theta}\right)S_{\alpha-\theta}}{-2\,I - m\,R^2 - 2\,M\,R^2 + m\,R^2\,C_{2(\alpha-\theta)}} \\ \frac{4\,m\,R\left(g + \dot\alpha^2 R - 2\,\dot\alpha\,\dot\theta\,R + \dot\theta^2 R\right)S_{\frac{\alpha-\theta}{2}}{}^2\,S_{\alpha-\theta}}{2\,I + m\,R^2 + 2\,M\,R^2 - m\,R^2\,C_{2(\alpha-\theta)}} + \\ -\frac{g\,m\,R\left(I + M\,R^2 + 4\,m\,R^2\,S_{\frac{\alpha-\theta}{2}}{}^2\right)S_{\alpha-\theta}}{I\,m\,R^2 + m^2\,R^4 + m\,M\,R^4 - m^2\,R^4\,C_{\alpha-\theta}{}^2} \end{bmatrix},$$

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ \frac{4\,S_{\frac{\alpha-\theta}{2}}{}^2\,\frac{R}{r}}{2\,I + m\,R^2 + 2\,M\,R^2 - m\,R^2\,C_{2(\alpha-\theta)}} \\ \frac{I + M\,R^2 + 4\,m\,R^2\,S_{\frac{\alpha-\theta}{2}}{}^2\,\frac{R}{r}}{I\,m\,R^2 + m^2\,R^4 + m\,M\,R^4 - m^2\,R^4\,C_{\alpha-\theta}{}^2}\} \end{bmatrix}$$

The approximate linearization of the equations of motion about the zero state is controllable (thus implying local controllability of the nonlinear model), and has two poles in zero, and two on the imaginary axis. In order to build stabilizing feedback controllers for this model, the measurement of the length traveled by the unicycle is sufficient. In fact, the linearized system is observable from an output $y = \alpha$, while, if only $\dot\alpha$ is available (as e.g. by odometry), states will be observable modulo the unobservable angle $\theta - \alpha$.

Remark 4. In some experimental setups it may be convenient to use stepper motors at the unicycle wheels. In such case, the dynamic model is changed by considering as input the acceleration rate of the stepper motors, i.e. $\ddot\alpha = u$, and observability from measurement of α is clearly lost. For this reason, in prototypes using stepper motors, a pendulum fixed to the unicycle can be added, whose angle measurements, together with α, can be shown to provide observability over the full system dynamics. Alternatively, an inclinometer or gyro can be employed.

Remark 5. The planar dynamic model of the sphericle provides an interesting case study for students to design and compare several different feedback controllers to stabilize the system, based on widely known linear feedback theory tools. Performances, robustness, computational issues, and applicability of the different methods can be impressively compared by laboratory experimentation. A far more advanced exercise is to obtain an expression of the dynamics of the sphericle in 3D, where nonholonomy comes to play, and to design stabilizing controllers in that case.

4 Planning

As discussed above, the sphericle lacks the structural properties that are required for application of the most powerful planning methods known in the literature. However, the sphericle motions can be planned by exploiting the triangular structure in (7), and the fact that the flows of the modified input vectorfields can be directly integrated as

$$\Phi_T^{\hat{\mathbf{g}}_1} =$$

$$\begin{bmatrix} x_0 + RC_{\theta_0}\left[\arcsin\sqrt{\frac{C^2_{v_0+T} - C^2_{\theta_0+\psi_0}C^2_{v_0}}{1 - C^2_{\theta_0+\psi_0}C^2_{v_0}}} - \arcsin\sqrt{\frac{C^2_{v_0}S^2_{\theta_0+\psi_0}}{1 - C^2_{\theta_0+\psi_0}C^2_{v_0}}}\right] \\ y_0 + RS_{\theta_0}\left[\arcsin\sqrt{\frac{C^2_{v_0+T} - C^2_{\theta_0+\psi_0}C^2_{v_0}}{1 - C^2_{\theta_0+\psi_0}C^2_{v_0}}} - \arcsin\sqrt{\frac{C^2_{v_0}S^2_{\theta_0+\psi_0}}{1 - C^2_{\theta_0+\psi_0}C^2_{v_0}}}\right] \\ u_0 - \left[\arcsin\sqrt{\frac{C^2_{\theta_0+\psi_0}C^2_{v_0}T^2_{v_0+T}}{1 - C^2_{\theta_0+\psi_0}C^2_{v_0}}} - \arcsin\sqrt{\frac{C^2_{\theta_0+\psi_0}S^2_{v_0}}{1 - C^2_{\theta_0+\psi_0}C^2_{v_0}}}\right] \\ v_0 + T \\ \theta_0 \\ \arccos\left[\frac{C_{\theta_0+\psi_0}C_{v_0}}{C_{v_0+T}}\right] - \theta_0 \end{bmatrix}$$

and $\Phi_T^{\hat{\mathbf{g}}_2} = [\ x_0 \quad y_0 \quad u_0 \quad v_0 \quad \psi_0 \quad \theta_0 + T\ ]$ Planning for the sphericle amounts therefore to solving the set of nonlinear equations

$$\left(\Phi_{Tk_6}^{\hat{\mathbf{g}}_2} \circ \Phi_{Tk_5}^{\hat{\mathbf{g}}_1} \circ \Phi_{Tk_4}^{\hat{\mathbf{g}}_2} \circ \Phi_{Tk_3}^{\hat{\mathbf{g}}_1} \circ \Phi_{Tk_2}^{\hat{\mathbf{g}}_2} \circ \Phi_{Tk_1}^{\hat{\mathbf{g}}_1}(\mathbf{x}_0) - \mathbf{x}_g\right) = 0$$

corresponding to the combined system flows corresponding to alternating inputs of the type

$$\begin{cases} \hat u_1 = k_1 \\ \hat u_2 = 0 \end{cases} \quad 0 < t < T$$
$$\begin{cases} \hat u_1 = 0 \\ \hat u_2 = k_2 \end{cases} \quad T < t < 2T$$
$$\vdots$$

Two examples of application of this planning technique are reported in fig.4 and fig.4, referring to different rotations of the ball obtained without changing the contact point positions and the orientation of the vehicle inside.

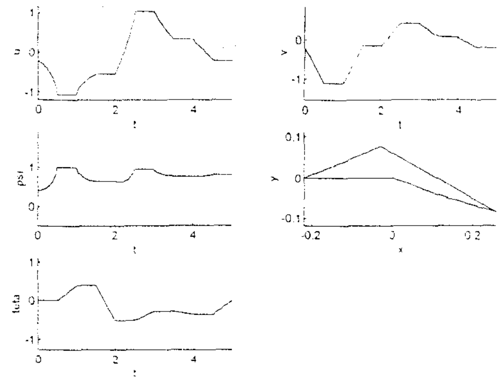

Figure 4: Path from $\mathbf{x}_0 = [0,\ 0,\ -\frac{\pi}{8},\ -\frac{\pi}{8},\ \frac{\pi}{8},\ 0]$ to $\mathbf{x}_0 = [0,\ 0,\ -\frac{\pi}{8},\ -\frac{\pi}{8},\ \frac{\pi}{4},\ 0]$. The plot in $x - -y$ coordinates describes the path followed by the vehicle in the plane.

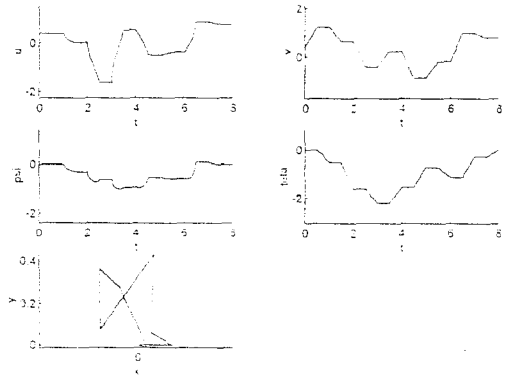

Figure 5: Path from $\mathbf{x}_0 = [0,\ 0,\ \frac{\pi}{8},\ \frac{\pi}{8},\ 0,\ 0]$ to $\mathbf{x}_0 = [0,\ 0,\ \frac{\pi}{4},\ -\frac{\pi}{4},\ 0,\ 0]$.

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