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# Manipulability of Cooperating Robots with Passive Joints

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## Abstract

*In this paper we study the differential kinematics and the kineto-static manipulability indices of multiple cooperating robot arms, including active and passive joints. The kinetic manipulability indices are derived as a simple extension of previous results on cooperating robots without passive joints. The force manipulability analysis for cooperative robot systems can not be derived by "duality" arguments as it can with conventional arms. rather a distinction between active and passive force manipulability is necessary. Results in this paper apply directly to the analysis of simply closed kinematic chains, and can be extended to mutiply closed kinematic chains.*

## 1 Introduction

The basic idea of manipulability analysis [9, 10] consists of describing directions in the joint space that extremize the ratio between some measure of effort in joint space, with a measure of performance in task space. Whenever these measures are quadratic functions of the joint and task variables, respectively, manipulability analysis amounts to the analysis of solutions to an eigenvalue–eigenvector problem [10].

The extension of manipulability analysis to multiple cooperating robots has been studied by several authors so far. Lee [5] and Chiacchio *et al.* [3] proposed extensions for the case when all cooperating arms have full mobility in their task space. Bicchi *et al.* [1] extended the kinetic manipulability ellipsoid problem to general cooperating arms, with arbitrary number of joints per arm. Park and Kim [6] studied manipulability of closed chains, including unactuated joints, using an elegant formulation in terms of differential geometric language. Bicchi, Prattichizzo, and Melchiorri [2] discussed the force manipulability problem for general cooperating arms with elasticity at joints and at contacts.

It is to be noted that the kinematics and statics of cooperating robot arms, including free kinematic pairs (such as a rolling or sliding contact, or an unactuated joint). and of closed kinematic chains, can be analyzed in a unified framework. The latter is a very important subject in mechanism design, and as such the mathematical tools proposed in this paper may have an impact on a very wide application domain.

In this paper we first show how several closed-chain problems can be solved by use of the formulation given in [1]. For cases when this is not possible, we introduce a generalization of those methods, that applies to general closed chain systems (section 2). Next, we discuss the problem of force manipulability for multiple arm/closed chain rigid systems, and show that a distinction between active and passive manipulability is in order to obtain physically meaningful indices (section 4). We conclude the paper by illustrating our results with some numeric examples (section 5).

## 2 Problem Formulation

The approach we follow to analyze kinematics and statics of closed-chain mechanical system is to consider them as embodiments of a cooperative manipulation paradigm, where multiple robotic *limbs* (or fingers) interact with an *object* at a number of *contacts*. The object is the reference member of the mechanism, whose motions are the ultimate goal of analysis. Contacts represent in fact unactuated kinematic pairs of different nature between the object and the contacting link. that restrict some or all the components of the relative velocities of the two bodies.

In [1], a notation for describing such systems was established which is recalled in the appendix. In that paper, each limb is allowed an arbitrary number of joints. Contacts with the object are allowed at any link of the various limbs. Fig. 1 shows how a four bar linkage with the two middle joints not actuated (a), can be thought of as a system (b) of two cooperating fingers and an object, with two contacts of bilateral soft-finger type.[1] For more general cases, where the unactuated joints are not all adjacent to one element of the chain, or when that element is not the member whose motions should be studied, methods of [1] have to be extended as de-

[1] Soft-finger contacts prevent all relative linear velocities, and allow relative angular velocities in the plane of contact, see the appendix.

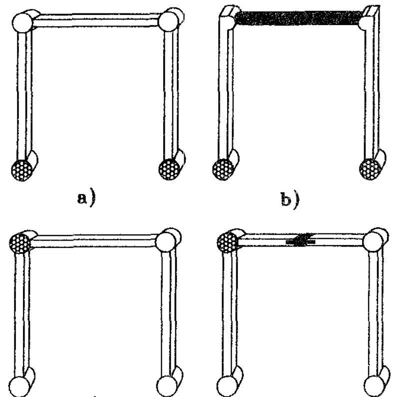


Figure 1: The four–bar linkage (a) with unactuated joints (in white) adjacent to the reference member can be considered as a manipulation system (b) with two one–joint fingers, one object (in black), and two soft–finger contacts. More generally, the four–bar linkage (c) can be represented as a manipulation system with two two–joint fingers, one object (in black), and two complete–constraint contacts.

scribed in the rest of this paper. Consider a system of cooperating robotic limbs, comprised of $q_a$ actuated joints, and $q_p$ unactuated (passive) simple kinematic joints, which interact with an object at $n$ contact points according to contact models as specified by a selection matrix $\mathbf{H}$ (see appendix). Let the aggreated Jacobian matrix of the cooperating devices be denoted $\tilde{\mathbf{J}}$, and let the object Jacobian (or grasp matrix) be $\tilde{\mathbf{G}}$. A suitable permutation matrix $\mathbf{P}$ can be found that reorders joint variables $\mathbf{q}$ to have actuated joints on top as

$$\dot{\mathbf{q}}' = \begin{bmatrix} \dot{\mathbf{q}}_a \\ \dot{\mathbf{q}}_p \end{bmatrix} = \mathbf{P}\dot{\mathbf{q}}, \qquad (1)$$

where $\mathbf{q}_a$ is the $q_a$–vector of actuated joints while $\mathbf{q}_p$ represents the $q_p$–vector of unactuated joints. Correspondingly, the Jacobian matrix can be partitioned as

$$\tilde{\mathbf{J}}\dot{\mathbf{q}} = \tilde{\mathbf{J}}\mathbf{P}^{-1}\dot{\mathbf{q}}' = \begin{bmatrix} \tilde{\mathbf{J}}_a & \tilde{\mathbf{J}}_p \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_a \\ \dot{\mathbf{q}}_p \end{bmatrix}.$$

The mobility of the system is then studied [1] by analyzing the constraint equation

$$\begin{bmatrix} \mathbf{J}_a & \mathbf{J}_p & -\mathbf{G}^T \end{bmatrix} \begin{bmatrix} \mathbf{q}_a \\ \mathbf{q}_p \\ \mathbf{u} \end{bmatrix} = \mathbf{0}. \qquad (2)$$

where $\mathbf{J}_a = \mathbf{H}\tilde{\mathbf{J}}_a$, $\mathbf{J}_p = \mathbf{H}\tilde{\mathbf{J}}_p$ and $\mathbf{G} = \mathbf{H}\tilde{\mathbf{G}}^T$. All possible motions of the system belong to the nullspace of the constraint matrix $\begin{bmatrix} \mathbf{J}_a & \mathbf{J}_p & -\mathbf{G}^T \end{bmatrix}$, and hence can be rewitten as linear combinations of vectors forming a basis of the nullspace. By suitable linear algebra operations, such a basis can always be written in a block partitioned form:

$$\begin{bmatrix} \dot{\mathbf{q}}_a \\ \dot{\mathbf{q}}_p \\ \dot{\mathbf{u}} \end{bmatrix} = \left[\begin{array}{c|cc} \mathbf{\Gamma}_{a,r} & \mathbf{\Gamma}_{a,c} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{\Gamma}_{pu,c} & \mathbf{\Gamma}_i \end{array}\right] \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix}. \qquad (3)$$

The generic rigid motion $\left[\dot{\mathbf{q}}_a^T \ \dot{\mathbf{q}}_p^T \ \dot{\mathbf{u}}^T\right]^T$ belonging to the nullspace of the constraint matrix is therefore parameterized by the components of vector $\left[\mathbf{x}_1^T \ \mathbf{x}_2^T \ \mathbf{x}_3^T\right]^T$.

In (3), $\mathbf{\Gamma}_{a,r}$ is a basis matrix of $\ker(\mathbf{J}_a)$ and incorporates the redundancy of the actuated part of the mechanism. All possible rigid–body motions of the actuated joints when both the reference member and the passive joints are locked can be written as linear combinations of columns of $\mathbf{\Gamma}_{a,r}$. Conversely, $\mathbf{\Gamma}_i = \ker\left[\mathbf{J}_p, \ -\mathbf{G}^T\right]$ represents the *kinetic indeterminacy* of the system, that is, all possible motions of the system, corresponding to actuated joints, locked belong to the column space of $\mathbf{\Gamma}_i$. The second block column of the matrix in (3) characterize the coordinate motions of the system. Vectors $\mathbf{\Gamma}_{pu,c}\mathbf{x}_2$ represent the unique possible motion of the object and of the passive joints corresponding to actuated joint motions $\mathbf{\Gamma}_{a,c}\mathbf{x}_2$.

A finer partition of block matrices in (3) can provide a more detailed analysis of mobility of systems under consideration. By algebraic manipulation, the motion indeterminacy matrix can be rewritten as

$$\mathbf{\Gamma}_i\mathbf{x}_3 = \begin{bmatrix} \mathbf{\Gamma}_{p,r} & \mathbf{\Gamma}_{pu,i} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}_{up,i} & \mathbf{\Gamma}_{u,i} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{31} \\ \mathbf{x}_{32} \\ \mathbf{x}_{33} \end{bmatrix}. \qquad (4)$$

Here, $\mathbf{\Gamma}_{p,r} = \ker(\mathbf{J}_p)$ incorporates all free motions of passive joints with both active joints and reference member (object) locked (as e.g. in a Stewart platform whose legs can rotate freely about the spherical joints at their extremities). On the other hand, $\mathbf{\Gamma}_{u,i} = \ker(\mathbf{G}^T)$ represents motions of the reference member that are not constrained when all joints, active and passive, are locked (this type of motion is usually avoided by design). Passive joint motions in the image of $\mathbf{\Gamma}_{pu,i}$ correspond one–to–one to object motions in the image of $\mathbf{\Gamma}_{up,i}$ when active joints are locked.

Similarly, one can rewrite

$$\left[\begin{array}{c} \mathbf{\Gamma}_{a,c} \\ \hline \mathbf{\Gamma}_{up,c} \end{array}\right] \mathbf{x}_2 = \left[\begin{array}{ccc} \mathbf{\Gamma}_{ap,c} & \mathbf{\Gamma}_{apu,c} & \mathbf{\Gamma}_{au,c} \\ \hline \mathbf{\Gamma}_{pa,c} & \mathbf{\Gamma}_{pua,c} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}_{upa,c} & \mathbf{\Gamma}_{ua,c} \end{array}\right] \begin{bmatrix} \mathbf{x}_{21} \\ \mathbf{x}_{22} \\ \mathbf{x}_{23} \end{bmatrix} \qquad (5)$$

to put into evidence coordinate motions of the actuated and unactuated joints while the object is locked (first block column), or coordinate motions of actuated joints and object with locked unactuated joints (third block column), and finally motions that are only possible by simultaneous movement of all joints and object.

As an example, consider the four-bar linkage of fig.1 (c), where the only actuated joint is in the middle of the chain. An equivalent manipulation system is depicted in (d), for the case when the reference member of interest in the linkage is its middle link. The system is comprised of two limbs with two joints, three of which are unactuated (in white), and an object (in black) grasped by two completely constraining contacts.

The method of analysis of closed kinematic chains based on the cooperative robots paradigm as described in this section can be applied to any simply closed kinematic chain, i.e. to any chain such that all closed loops can be broken by removing a single member, in the case that this member is also the member of reference. Thus, a Stewart platform can be regarded as a system of six legs, each with three unactuated joints (forming a spherical joints at the base of the leg) and one actuated prismatic joint in their middle, all legs being in contact with the platform (object) by hard-finger contacts. For more general mechanisms, e.g. multiple kinematic chains without a common element breaking all loops, more than one "object" should be considered.

## 3 Kinematic Manipulability

A kinematic manipulability index can be defined in terms of the ratio of a measure of performance in the task space and a measure of effort in the joint space. Taking these measures to be suitably defined norms of velocities, an index can be written as

$$R_v = \frac{\dot{\mathbf{u}}^T \mathbf{W}_u \dot{\mathbf{u}}}{\dot{\mathbf{q}}^T \mathbf{W}_q \dot{\mathbf{q}}}, \tag{6}$$

where $\mathbf{W}_u$, $\mathbf{W}_q$ are positive definite matrices whose role is to weight different components of velocities in the two spaces (including the case of nonhomogeneous units for linear or angular velocities). Observe that this choice of weights effectively amounts to defining a metric on the tangent space to the task and joint manifolds [6]. In practice, the choice of $\mathbf{W}_q$ is made based on how much it "costs" to run a certain actuator at unit velocity. The choice of $\mathbf{W}_u$ is usually made based on the task specifications (see e.g. [11]); however note that a physically motivated choice could be taking $\mathbf{W}_u$ as the inertia matrix of the reference member.[2]

In [1], it was shown how ratio (6) effectively incorporates the traditional maipulability of serial–chain manipulators, and it was extended to encompass multiple–limb manipulation systems. This section shows how those results can be extended to the case of unactuated joints. Being interested to performance in the space of velocities of the reference member $\dot{\mathbf{u}}$, and efforts in the space of actuated joints, the index is rewritten as

$$R_{va} = \frac{\dot{\mathbf{u}} \mathbf{W}_u \dot{\mathbf{u}}}{\dot{\mathbf{q}}_a \mathbf{W}_u \dot{\mathbf{q}}_a}. \tag{7}$$

The analysis of kinematic efficiency, providing information about which directions in the task space (and corresponding directions in the actuated joint space) maximize or minimize $R_{va}$, is easily solved once a correspondence between the numerator and denominator variables, namely $\dot{\mathbf{u}}$ and $\dot{\mathbf{q}}_a$, in (7), , is established. Note that, in order for the ratio (6) to be well–defined, a one–to–one mapping should be established between the two variables.

To find such mapping, consider kinematic relationships (3), (4) and (5), which can be rewritten as

$$\begin{cases} \dot{\mathbf{q}}_a & = \mathbf{\Gamma}_{a,r}\mathbf{x}_1 + \mathbf{\Gamma}_{ap,c}\mathbf{x}_{21} + \mathbf{\Gamma}_{apu,c}\mathbf{x}_{22} + \mathbf{\Gamma}_{au,c}\mathbf{x}_{23} \\ & = \mathbf{\Gamma}_r\mathbf{x}_r + \mathbf{\Gamma}_{q,c}\mathbf{x}_c; \\ \dot{\mathbf{u}} & = \mathbf{\Gamma}_{upa,c}\mathbf{x}_{22} + \mathbf{\Gamma}_{ua,c}\mathbf{x}_{23} + \mathbf{\Gamma}_{up,i}\mathbf{x}_{32} + \mathbf{\Gamma}_{u,i}\mathbf{x}_{33} \\ & = \mathbf{\Gamma}_{u,c}\mathbf{x}_c + \mathbf{\Gamma}_{ii}\mathbf{x}_i; \end{cases} \tag{8}$$

where $\mathbf{\Gamma}_r = [\mathbf{\Gamma}_{a,r}\ \mathbf{\Gamma}_{ap,c}]$, $\mathbf{\Gamma}_{q,c} = [\mathbf{\Gamma}_{apu,c}\ \mathbf{\Gamma}_{au,c}]$, $\mathbf{\Gamma}_{u,c} = [\mathbf{\Gamma}_{upa,c}\ \mathbf{\Gamma}_{ua,c}]$ and $\mathbf{\Gamma}_{ii} = [\mathbf{\Gamma}_{up,i}\ \mathbf{\Gamma}_{u,i}]$.

From (8) it appears that a one–to–one relationship between task and acuated joints velocities does not exist in general, because of the possible presence of redundancy (matrix $\mathbf{\Gamma}_r$) and indeterminacy (matrix $\mathbf{\Gamma}_i$). However, based on the physical meaning of the efficiency ratio, it is reasonable to assume that, if more than one actuated joint velocity can be chosen corresponding to some task velocity, then the one with minimum cost will be preferred.

The case with system indeterminacy is of far less interest in practice, as systems motions can not be predicted based on a kinematic model only. However, taking a conservative point of view, it is considered that if there are indeed more than one possible velocities of the object corresponding to the same joint velocity, the efficiency will never be worse than the case where the object moves so as to minimize the numerator[3].

Taking this into account, and using (8), we redefine the efficiency ratio (7) as the worst–case (numerator), optimized (denominator) efficiency ratio

$$R_{va}^{ow} = \frac{\min_{\mathbf{x}_i}\ \dot{\mathbf{u}} \mathbf{W}_u \dot{\mathbf{u}}}{\min_{\mathbf{x}_r}\ \dot{\mathbf{q}}_a \mathbf{W}_u \dot{\mathbf{q}}_a}. \tag{9}$$

Using some standard linear algebraic tools such as pseudoinverses and projectors, reported in appendix B,

[2] In this case, the numerator of (6) would represent twice the kinetic energy of the object.

[3] If the object inertia matrix is chosen as weight, this corresponds also to assume that the actual object velocity will minimize its kinetic energy.

the two minimization problems can be easily solved as

$$\min_{\mathbf{x}_c} \dot{\mathbf{u}}^T \mathbf{W}_u \dot{\mathbf{u}} = \mathbf{x}_c^T \mathbf{\Gamma}_{uc}^T \mathbf{P}_{(\mathbf{\Gamma}_{ii}^*, W_u)}^T \mathbf{W}_u \mathbf{P}_{(\mathbf{\Gamma}_{ii}^*, W_u)} \mathbf{\Gamma}_{uc} \mathbf{x}_c.$$
$$\min_{\mathbf{x}_c} \dot{\mathbf{q}}^T \mathbf{W}_q \dot{\mathbf{q}} = \mathbf{x}_c^T \mathbf{\Gamma}_{qc}^T \mathbf{P}_{(\mathbf{\Gamma}_r^*, W_q)}^T \mathbf{W}_q \mathbf{P}_{(\mathbf{\Gamma}_r^*, W_q)} \mathbf{\Gamma}_{qc} \mathbf{x}_c.$$

Therefore, the (optimized, worst-case) kinematic manipulability analysis is reduced to studying the ratio

$$R_{va}^{ow} = \frac{\mathbf{x}_c^T \mathbf{\Gamma}_{uc}^T \mathbf{P}_{(\mathbf{\Gamma}_{ii}^*, W_u)}^T \mathbf{W}_u \mathbf{P}_{(\mathbf{\Gamma}_{ii}^*, W_u)} \mathbf{\Gamma}_{uc} \mathbf{x}_c}{\mathbf{x}_c^T \mathbf{\Gamma}_{qc}^T \mathbf{P}_{(\mathbf{\Gamma}_r^*, W_q)}^T \mathbf{W}_q \mathbf{P}_{(\mathbf{\Gamma}_r^*, W_q)} \mathbf{\Gamma}_{qc} \mathbf{x}_c}$$

at varying $\mathbf{x}_c$, i.e. a standard generalized eigenvalue problem [4]. The maximum value of $R_{va}^{ow}$ corresponds to the maximum generalized eigenvalue $\lambda_{max}$ solving

$$\mathbf{N}\mathbf{x} = \lambda \mathbf{D}\mathbf{x}.$$

or, equivalently, the eigenvalue problem $\mathbf{D}^{-1/2}\mathbf{N}\mathbf{D}^{-1/2}\mathbf{x} = \lambda \mathbf{x}$, with

$$\begin{aligned} \mathbf{N} &= \mathbf{\Gamma}_{uc}^T \mathbf{P}_{(\mathbf{\Gamma}_{ii}^*, W_u)}^T \mathbf{W}_u \mathbf{P}_{(\mathbf{\Gamma}_{ii}^*, W_u)} \mathbf{\Gamma}_{uc}; \\ \mathbf{D} &= \mathbf{\Gamma}_{qc}^T \mathbf{P}_{(\mathbf{\Gamma}_r^*, W_q)}^T \mathbf{W}_q \mathbf{P}_{(\mathbf{\Gamma}_r^*, W_q)} \mathbf{\Gamma}_{qc}. \end{aligned}$$

The corresponding generalized eigenvector $\mathbf{x}_{max}$ gives the direction, in the parameter space, where maximum performance is obtained. The corresponding directions in the task and joint velocity spaces are, respectively,

$$\begin{aligned} \dot{\mathbf{u}}_{max} &= \mathbf{P}_{(\mathbf{\Gamma}_{ii}^*, W_u)} \mathbf{\Gamma}_{uc} \mathbf{x}_{max}; \\ \dot{\mathbf{q}}_{max} &= \mathbf{P}_{(\mathbf{\Gamma}_r^*, W_q)} \mathbf{\Gamma}_{qc} \mathbf{x}_{max}. \end{aligned} \tag{10}$$

Obviously, similar considerations apply for $\lambda_{min}$, the minimum generalized eigenvalue, and $\mathbf{x}_{min}$, the corresponding eigenvector.

## 4 Force manipulability ellipsoids

The force manipulability index is defined in a manner similar to kinematic manipulability as the ratio of a performance measure in the space of forces exchanged with the environment, and an effort measure in the space of (actuated) joint torques:

$$R_{fa} = \frac{\mathbf{w}^T \mathbf{W}_w \mathbf{w}}{\tau^T \mathbf{W}_\tau \tau}. \tag{11}$$

Here, weights in $\mathbf{W}_\tau$ incorporate different costs in generating torque or forces at joints, and takes care of mismatches of measurement units between rotoidal and prismatic joints. Weights in $\mathbf{W}_w$ adjust for different units of components of wrench $\mathbf{w}$, and may represent task specifications (such as greater leverage in some direction). A physically motivated choice could be taking $\mathbf{W}_w$ as the stiffness matrix of the environment with wich the reference member interacts.[4]

[4] in this case, the numerator of (11) would represent twice the elastic energy of interaction.

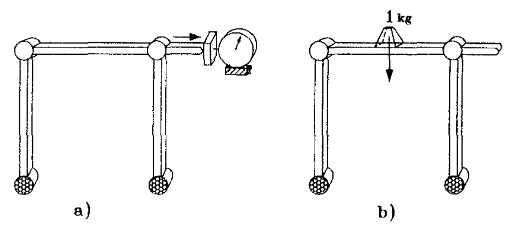


Figure 2: A cooperating system may actively exert a wrench (a) and passively resist an external wrench (b) with best efficiency in different directions.

The relation between wrenches on the reference member $\mathbf{w}$ and actuated joint torques $\tau_a$ at equilibrium follows from application of the virtual work principle as

$$\begin{aligned} \mathbf{w} &= \mathbf{G}\mathbf{t} \\ \tau_a &= -\mathbf{J}_a^T \mathbf{t} \\ \mathbf{0} &= -\mathbf{J}_p^T \mathbf{t} \end{aligned} \tag{12}$$

where $\mathbf{t}$ is a $t$-dimensional vector of contact forces. By rewriting these equations in matrix form as

$$\begin{bmatrix} \mathbf{I} & -\mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_a^T & \mathbf{I} \\ \mathbf{0} & \mathbf{J}_p^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{t} \\ \tau_a \end{bmatrix} = \mathbf{0}$$

and assuming that the system is not hyperstatic, i.e., $\ker(\mathbf{G}) \cap \ker(\mathbf{J}_a^T) = \{0\}$ [7], it follows that all equilibrium combinations of external wrenches and active joint torques can be written as [8]

$$\begin{bmatrix} \mathbf{w} \\ \tau_a \end{bmatrix} = \left[ \begin{array}{c|c|c} \mathbf{0} & \mathbf{\Gamma}_w & \mathbf{\Gamma}_s \\ \mathbf{\Gamma}_h & \mathbf{\Gamma}_\tau & \mathbf{0} \end{array} \right] \begin{bmatrix} \mathbf{x}_h \\ \mathbf{x}_a \\ \mathbf{x}_s \end{bmatrix}. \tag{13}$$

Here, $\mathbf{\Gamma}_h$, $\mathbf{\Gamma}_s$, $\mathbf{\Gamma}_w$, $\mathbf{\Gamma}_\tau$, and $\mathbf{\Gamma}_t$ are obtained by an algorithm similar to that used in 2 and described in [1].

In [2] the authors introduced a distinction between active and passive force manipulability motivated for instance by observing the simple systems described in fig.2. It appears that wrenches which the manipulation system is able to apply most efficiently through the object to the environment, differ from wrenches that are most efficently resisted if external loads act on the object. It is then natural to introduce two force manipulability indices, for the active and passive cases.

### 4.1 Active force manipulability

For a given set of equilibrium torques at the actuated joints, i.e., for given $\mathbf{x}_a$ and $\mathbf{x}_h$ in (13), the corresponding wrench is not uniquely defined if a nullspace of $\mathbf{J}^T = [\mathbf{J}_a \ \mathbf{J}_p]^T$ exists. However, the worst–case efficiency will be given by

$$R_{af} = \frac{\min_{x_s} \ \mathbf{w}^T \mathbf{W}_w \mathbf{w}}{\tau_a^T \mathbf{W}_\tau \tau_a}. \qquad (14)$$

Note that, if $\mathbf{W}_w$ takes into account the environmental stiffness, minimization of the numerator amounts to assuming that the mechanism apply, for the given joint torques, the wrench that minimizes the energy of elastic deformation. By using results of Appendix B, one readily gets

$$\min_{\mathbf{x}_s} \mathbf{w}^T \mathbf{W}_w \mathbf{w} = \mathbf{x}_a^T \mathbf{\Gamma}_w^T \mathbf{P}^T_{(\mathbf{\Gamma}_s^*, W_u)} \mathbf{W}_u \mathbf{P}_{(\mathbf{\Gamma}_s^*, W_u)} \mathbf{\Gamma}_w \mathbf{x}_a.$$

Therefore, the worst–case active force manipulability analysis is reduced to studying the ratio

$$R_{ra}^{ow} = \frac{\begin{bmatrix} \mathbf{x}_a \\ \mathbf{x}_h \end{bmatrix}^T \left[\begin{array}{c|c} \mathbf{\Gamma}_w^T \mathbf{P}^T_{(\mathbf{\Gamma}_s^*, W_u)} \mathbf{W}_u \mathbf{P}_{(\mathbf{\Gamma}_s^*, W_u)} \mathbf{\Gamma}_w & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] \begin{bmatrix} \mathbf{x}_a \\ \mathbf{x}_h \end{bmatrix}}{\begin{bmatrix} \mathbf{x}_a \\ \mathbf{x}_h \end{bmatrix}^T \left[\begin{array}{c|c} \mathbf{\Gamma}_\tau^T \mathbf{W}_\tau \mathbf{\Gamma}_\tau & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{\Gamma}_h^T \mathbf{W}_\tau \mathbf{\Gamma}_h \end{array}\right] \begin{bmatrix} \mathbf{x}_a \\ \mathbf{x}_h \end{bmatrix}}, \qquad (15)$$

i.e. a standard generalized eigenvalue problem The discussion of the ellipsoid is similar to the one given above for kinematic manipulability. Note that the numerator quadratic form has a number of zero eigenvalues equal to the components of $\mathbf{x}_h$, corresponding to joint torques balanced by purely internal contact forces, with non net effect on the object, that obviously give zero efficiency[5]

## 4.2 Passive force manipulability

For a given equilibrium wrench acting externally on the reference member, i.e., for given $\mathbf{x}_a$ and $\mathbf{x}_s$ in (13), the corresponding joint torques are not uniquely defined if a nullspace of $\mathbf{G}$ (internal forces) exists. However, it is reasonable to assume that the torque with minimum cost will be chosen to oppose the wrench. The optimized passive force efficiency will be given by

$$R_{af} = \frac{\mathbf{w} \mathbf{W}_w \mathbf{w}}{\min_{x_h} \ \tau_a^T \mathbf{W}_\tau \tau_a}.$$

According to appendix B, one gets

$$\min_{x_h} \ \tau_a^T \mathbf{W}_\tau \tau_a = \mathbf{x}_a^T \mathbf{\Gamma}_\tau^T \mathbf{P}^T_{(\mathbf{\Gamma}_h^*, W_\tau)} \mathbf{W}_\tau \mathbf{P}_{(\mathbf{\Gamma}_h^*, W_\tau)} \mathbf{\Gamma}_\tau \mathbf{x}_a,$$

and the optimized passive force manipulability analysis is studied by the ratio

$$R_{va}^{ow} = \frac{\begin{bmatrix} \mathbf{x}_a \\ \mathbf{x}_s \end{bmatrix}^T \left[\begin{array}{c|c} \mathbf{\Gamma}_w^T \mathbf{W}_w \mathbf{\Gamma}_w & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{\Gamma}_s^T \mathbf{W}_u \mathbf{\Gamma}_s \end{array}\right] \begin{bmatrix} \mathbf{x}_a \\ \mathbf{x}_s \end{bmatrix}}{\begin{bmatrix} \mathbf{x}_a \\ \mathbf{x}_s \end{bmatrix}^T \left[\begin{array}{c|c} \mathbf{\Gamma}_\tau^T \mathbf{P}^T_{(\mathbf{\Gamma}_h^*, W_\tau)} \mathbf{W}_\tau \mathbf{P}_{(\mathbf{\Gamma}_h^*, W_\tau)} \mathbf{\Gamma}_\tau & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] \begin{bmatrix} \mathbf{x}_a \\ \mathbf{x}_s \end{bmatrix}}. \qquad (16)$$

[5] Internal forces are indeed useful for manipulation. Internal force manipulability indices may also have to be considered, but extension to this case of our approach is easy and omitted for space limitations

Note that the denominator quadratic form has a number of zero eigenvalues equal to the components of $\mathbf{x}_s$, corresponding to wrenches balanced by structural constraints, with no net effect on the active joints (nor on the passive, because of the equilibrium condition (12), that obviously give infinite efficiency.

## 4.3 Duality

From the treatment of preceding sections, the usual duality relationship between kinematic and force ellipsoids is somewhat concealed. In fact, it is true that in practice, for manipulation systems such as those considered here, the kinematic and force domains have differences: for instance, while the existence of $\ker(\mathbf{G})$ (internal forces) and $\ker(\mathbf{J}^T)$ (zero–torque forces) is the norm in practical devices, existence of a redundancy subspace $\ker(\mathbf{J})$ is not so frequent, and systems with non–trivial $\ker(\mathbf{G}^T)$, (contact) indeterminacy, are rare. This explains why the two domains have been treated differently in the above sections.

However, for the sake of completeness, it should be mentioned that it is indeed possible to define an efficiency index of active kinematic manipulability as

$$R_{ak} = \frac{\min_{\mathbf{x}_i} \ \dot{\mathbf{u}} \mathbf{W}_u \dot{\mathbf{u}}}{\dot{\mathbf{q}}_a \mathbf{W}_u \dot{\mathbf{q}}_a},$$

and one for passive kinematic manipulability as

$$R_{pk} = \frac{\dot{\mathbf{u}} \mathbf{W}_u \dot{\mathbf{u}}}{\min_{\mathbf{x}_r} \ \dot{\mathbf{q}}_a \mathbf{W}_u \dot{\mathbf{q}}_a},$$

which will have a number of zero eigenvalues equal to the dimensions of the redundancy subspace, and a number of infinite eigenvalues equal to the dimension of the indeterminacy subspace, respectively. A physical interpretation of $R_{ak}$ is the worst–case efficiency for given joint velocities, while $R_{pk}$ can be thought of as a kinematic manipulability when the mechanism is actuated *from* the object, and velocities at the active joints are considered as outputs. In the force domain, a worst–case optimized efficiency index can be defined as

$$R_{fa}^{ow} = \frac{\min_{x_s} \mathbf{w}^T \mathbf{W}_w \mathbf{w}}{\min_{x_h} \ \tau_a^T \mathbf{W}_\tau \tau_a}. \qquad (17)$$

If problems (9) and (17) are compared, with $\mathbf{W}_u \mathbf{W}_w = \mathbf{W}_q \mathbf{W}_\tau$ and taking into account that, because of the principle of virtual work, it holds $\mathbf{\Gamma}_{uc}^T \mathbf{\Gamma}_w = \mathbf{\Gamma}_{qc}^T \mathbf{\Gamma} \tau$, $\mathbf{\Gamma}_\tau^T \mathbf{\Gamma}_r = \mathbf{\Gamma}_{qc}^T \mathbf{\Gamma} h = \mathbf{\Gamma}_i^T \mathbf{\Gamma}_w = \mathbf{\Gamma}_s^T \mathbf{\Gamma}_{uc} = 0$, it is found that the two generalized eigenvalue problems are equal (both the numerator and denominator forms are the same). This involves that directions in which the manipulator moves most efficiently (once redundancy is optimized) are also directions in which forces are applied best (with optimized internal forces) through the object on the environment.

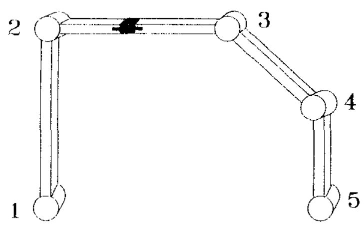


Figure 3: A five–bar linkage used as a case study. Case a): only joints 1 and 5 actuated; case b): only joints 2 and 4 actuated; case c): all joints actuated.

## 5 Examples

As a case study, consider a two–arm system forming a five–bar closed chain (see fig.3). The system of fig.3 has no indeterminacy nor redundancy, while the dimension of the subspace of internal forces is 3, and the dimension of the subspace of zero–torque wrenches is 1.

If actuated joints are 1 and 5 only (case a), the two generalized eigenvalues of the kinematic manipulability ellipsoid are 13.4 and 0.6, while, if only joints 2 and 4 are actuated (case b), they evaluate to 1.55 and 0.32. If all joints are actuated (case c), the eigenvalues are 0.63 and 0.17.

In the force domain, the active force ellipsoid for case (a) has two eigenvalues at 1.68 and 0.07; case (b) gives 3.1 and 0.64, while in case (c) there are three zero eigenvalues (corresponding to the dimension of the internal forces), and two eigenvalues at 3.5 and 1.35.

The passive force ellipsoid for case (a) has one infinite eigenvalue (corresponding to an external wrench on the object equivalent to a force aligned with the first link), and two finite values at 1.7 and 0.08. In case (b) one has again one infinite eigenvalue corresponding to the same wrench direction, and finite eigenvalues at 3.3, and 0.7; case (c) gives two finite eigenvalues at 6.0 and 1.6, in addition to the same infinite eigenvalue as cases (b) and (c).

From the examples, it appears clearly how actuating inner joints of the 5–bar linkage produces a more isotropic ellipsoid under all regards, kinematic, active and passive force, and that the optimal condition under that regard is achieved when all joints are actuated.

## Appendix A: Notation

The quantities introduced in text are defined as follows. Let $s = 2, d = 3$ for 2D mechanisms, and $s = 3, d = 6$ for 3D ones. Moreover let $q = q_a + q_p$ be the number of actuated joints, $n$ the number of contacts, and set

$$\begin{aligned} \dot{\mathbf{q}} &= [\dot{q}_1, \dot{q}_2, \dots, \dot{q}_q]^T; \quad \dot{\mathbf{q}} \in \mathbb{R}^q; \\ \tau &= [\tau_1, \tau_2, \dots, \tau_q]^T; \quad \tau \in \mathbb{R}^q; \\ \dot{\mathbf{u}} &= [\mathbf{v}^T, \omega^T]^T; \quad \dot{\mathbf{u}} \in \mathbb{R}^d; \\ \mathbf{w} &= [\mathbf{f}^T, \mathbf{m}^T]^T; \quad \mathbf{w} \in \mathbb{R}^d, \end{aligned}$$

where $\mathbf{v}$ ($\omega$) is the linear (angular) velocity of object and $\mathbf{f}$ ($\mathbf{m}$) is the force (moment) on the object.

Denoting by $\mathbf{c}_i$ the position of the $i$–th contact point and by $\mathbf{p}$ the object center of mass, let

$$\tilde{\mathbf{G}} = \left[ \begin{array}{ccc|ccc} \mathbf{I}_s & \cdots & \mathbf{I}_s & & \mathbf{O}_{c\times ns} & \\ \mathbf{S}(\mathbf{c_1}-\mathbf{p}) & \cdots & \mathbf{S}(\mathbf{c_n}-\mathbf{p}) & \mathbf{I}_s & \cdots & \mathbf{I}_s \end{array} \right];$$

$$\tilde{\mathbf{J}}^T = \left[ \begin{array}{ccc|ccc} \mathbf{D}_{1,1} & \cdots & \mathbf{D}_{n,1} & \mathbf{L}_{1,1} & \cdots & \mathbf{L}_{n,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{D}_{1,r} & \cdots & \mathbf{D}_{n,r} & \mathbf{L}_{1,r} & \cdots & \mathbf{L}_{n,r} \end{array} \right];$$

where

$$\mathbf{S}(\mathbf{c_i}) = \begin{bmatrix} 0 & -c_{i,y} & c_{i,z} \\ c_{i,y} & 0 & -c_{i,x} \\ -c_{i,z} & c_{i,x} & 0 \end{bmatrix}, \quad for \ \ s = 3;$$

$$\mathbf{S}(\mathbf{c_i}) = \begin{bmatrix} -c_{i,y} & c_{i,x} \end{bmatrix}, \quad for \ \ s = 2;$$

blocks $\mathbf{D}_{i,j}$ and $\mathbf{L}_{i,j}$ are defined as

$$\mathbf{D}_{i,j} = \begin{cases} [0\ 0\ 0] & \text{if the } i\text{-th contact force does not affect the } j\text{-th joint;} \\ \mathbf{z}_j^T & \text{for prismatic } j\text{-th joint;} \\ \mathbf{z}_j'^T\ \mathbf{S}(\mathbf{c_i}-\mathbf{o_j}) & \text{for rotational } i\text{-th joint;} \end{cases}$$

$$\mathbf{L}_{i,j} = \begin{cases} [0\ 0\ 0] & \text{if the } i\text{-th contact force does not affect the } j\text{-th joint;} \\ [0\ 0\ 0] & \text{for prismatic } j\text{-th joint;} \\ \mathbf{z}_j'^T & \text{for rotational } j\text{-th joint;} \end{cases}$$

being $\mathbf{o}_j$ and $\mathbf{z}_j$ the center and $z$-axis unit vector of the Denavit-Hartenberg frames associated with the $j$-th joint while $\mathbf{z}'_j = \mathbf{z}_j$ if $s = 3$ and $\mathbf{z}'_j = 1$ if $s = 2$.

The column space of matrices $\tilde{\mathbf{G}}^T$ and $\tilde{\mathbf{J}}$ represent linear and angular velocities (in all directions) of frames attached to all contact points as a function of object and joint velocities respectively.

Rigid–body contact constraints of different types can be written as

$$\mathbf{H}(\tilde{\mathbf{J}}\dot{\mathbf{q}} - \tilde{\mathbf{G}}^T\dot{\mathbf{u}}) = \mathbf{J}\dot{\mathbf{q}} - \mathbf{G}^T\dot{\mathbf{u}} = 0$$

where the selection matrix $\mathbf{H}$ is built according to different contact models as reported in table 1. The overall contact selection matrix $\mathbf{H}$ is obtained by removing the zero rows from matrix

$$\hat{\mathbf{H}} = \mathrm{diag}(FS_1, \dots, FS_n, MS_1, \dots, MS_n).$$

Observe that conventional kinematic joints between a link and the object can be modelled as unactuated contacts. In this case, the differential kinematic (contact–like) constraints used to build the selection matrix are

| Contact Type | Force Selector $FS_i$ | Moment Selector $MS_i$ |
|---|---|---|
| Point Contact w/o Friction | $\mathbf{z}_i^T$ | $\mathbf{0}_{1 \times d-s}$ |
| Point Contact w/h Friction (Hard Finger) | $\mathbf{I}_s$ | $\mathbf{0}_{1 \times d-s}$ |
| Line Contact w/o Friction | $\mathbf{z}_i^T$ | $(S(\mathbf{z}_i)\mathbf{x}_i)^T$ |
| 3D Line Contact w/h Friction | $\mathbf{z}_i^T$ | $\begin{bmatrix} (S(\mathbf{z}_i)\mathbf{x}_i)^T \\ \mathbf{z}_i^T \end{bmatrix}$ |
| 3D Planar Contact w/o Friction | $\mathbf{z}_i^T$ | $\begin{bmatrix} \mathbf{x}_i^T \\ \mathbf{y}_i^T \end{bmatrix}$ |
| Planar Contact w/h Friction (Complete Constraint) | $\mathbf{I}_s$ | $\mathbf{I}_{d-s}$ |
| 3D Soft Finger | $\mathbf{I}_3$ | $\mathbf{z}_i^T$ |

Table 1: Selectors for different contact types used to build the selection matrix $\mathbf{H}$. Vector $\mathbf{z}_i$ is the unit surface normal at the $i$-th contact while $\mathbf{x}_i$ and $\mathbf{y}_i$ are two unit vectors defining the line and plane of contact.

reported in table 2. The possibility of describing unactuated joints as bilateral contact constraints enable us to formalize the analysis of closed kinematic chains in a cooperative robots framework, cf. Section 2.

| Joint Type | Force Selector $FS_i$ | Moment Selector $MS_i$ |
|---|---|---|
| 3D Rotoidal | $\mathbf{I}_3$ | $\begin{bmatrix} \mathbf{x}_i^T \\ \mathbf{y}_i^T \end{bmatrix}$ |
| 2D Rotoidal | $\mathbf{I}_2$ | $0$ |
| 3D Prismatic | $\begin{bmatrix} \mathbf{x}_i^T \\ \mathbf{y}_i^T \end{bmatrix}$ | $\mathbf{I}_3$ |
| 2D Prismatic | $\mathbf{x}_i^T$ | $1$ |
| 3D Spherical | $\mathbf{I}_3$ | $\mathbf{0}_{1 \times 3}$ |

Table 2: Selectors for different joint types used to build the selection matrix $\mathbf{H}$. Vectors $\mathbf{x}_i$ and $\mathbf{y}_i$ denote two unit vectors normal to the joint axis $\mathbf{z}_i$.

## Appendix B: Minimization problem

Consider the quadratic minimization problem

$$V(\hat{\mathbf{x}}) = \min_{\mathbf{x}} V(\mathbf{x}) = \min_{\mathbf{x}} \mathbf{x}^T \mathbf{Q} \mathbf{x} \qquad (18)$$

subject to the linear constraint

$$\mathbf{x} = \mathbf{A}\mathbf{y} + \mathbf{b}. \qquad (19)$$

where $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m, \mathbf{Q} \in \mathbb{R}^{n \times n}, \mathbf{b} \in \mathbb{R}^n$. $\mathbf{Q}$ is semi-definite positive definite and $\mathbf{A}$ is full column rank. Under the hypothesis that

$$\ker(\mathbf{Q}) \cap \mathrm{im}(\mathbf{A}) = \mathbf{0}. \qquad (20)$$

The solution is easily obtained by standard linear algebraic tools as follows. Problem (19) is equivalent to the unconstrained problem

$$\min_{\mathbf{y}} V(\mathbf{y}) = \min_{\mathbf{y}} (\mathbf{A}\mathbf{y} + \mathbf{b})^T \mathbf{Q} (\mathbf{A}\mathbf{y} + \mathbf{b}).$$

By setting $\frac{\partial V(\mathbf{y})}{\partial \mathbf{y}} = \mathbf{0}$ one gets $\mathbf{A}^T\mathbf{Q}\mathbf{A}\hat{\mathbf{y}} = -\mathbf{A}^T\mathbf{Q}\mathbf{b}$ and, because from (20) matrix $\mathbf{A}^T\mathbf{Q}\mathbf{A}$ is invertible,

$$\hat{\mathbf{y}} = -(\mathbf{A}^T\mathbf{Q}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Q}\mathbf{b} = -\mathbf{A}_Q^+\mathbf{b}.$$

Matrix $\mathbf{A}_Q^+$ is the $\mathbf{Q}$-weighted pseudoinverse of $\mathbf{A}$. Vector $\hat{\mathbf{y}}$ actually minimizes $V(\mathbf{y})$, in fact $\frac{\partial^2 V}{\partial \mathbf{y}^2} = \mathbf{A}^T\mathbf{Q}\mathbf{A}$ is positive definite. The optimizing vector of the original problem is

$$\hat{\mathbf{x}} = (\mathbf{I} - \mathbf{A}\mathbf{A}_Q^+)\mathbf{b} = \mathbf{P}_{(\mathbf{A}^*,Q)}\,\mathbf{b}.$$

Matrix $\mathbf{P}_{(\mathbf{A}^*,Q)}$ is the projector on $\ker(\mathbf{A})^T$ that minimizes the $\mathbf{Q}$-weighted length of the projected vector.

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