



## Brief paper

# A new approach to the cheap LQ regulator exploiting the geometric properties of the Hamiltonian system<sup>☆</sup>

Domenico Prattichizzo<sup>a,\*</sup>, Lorenzo Ntogramatzidis<sup>b</sup>, Giovanni Marro<sup>c</sup>

<sup>a</sup> Dipartimento di Ingegneria dell' Informazione, Università di Siena, via Roma 56 - I-53100, Italy

<sup>b</sup> Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, VIC 3010, Australia

<sup>c</sup> Dipartimento di Elettronica, Informatica e Sistemistica, Università di Bologna, viale Risorgimento, 2 – 40136 Bologna, Italy

## ARTICLE INFO

## Article history:

Received 10 April 2007

Received in revised form

15 November 2007

Accepted 22 February 2008

Available online 26 September 2008

## Keywords:

Geometric techniques

Cheap LQ problem

Hamiltonian system

Continuous-time systems

## ABSTRACT

The cheap LQ regulator is reinterpreted as an output nulling problem which is a basic problem of the geometric control theory. In fact, solving the LQ regulator problem is equivalent to keep the output of the related Hamiltonian system identically zero. The solution lies on a controlled invariant subspace whose dimension is characterized in terms of the minimal conditioned invariant of the original system, and the optimal feedback gain is computed as the friend matrix of the resolving subspace. This study yields a new computational framework for the cheap LQ regulator, relying only on the very basic and simple tools of the geometric approach, namely the algorithms for controlled and conditioned invariant subspaces and invariant zeros.

© 2008 Elsevier Ltd. All rights reserved.

## 1. Introduction

It is well known that the solution of the infinite-horizon linear quadratic (LQ) regulator problem strongly depends on the matrix weighting the input in the cost function, traditionally denoted by  $R$ . When  $R$  is positive definite, the problem is said to be *regular* (see e.g., Anderson and Moore (1989), Lewis and Syrmos (1995) and Ionescu, Oară and Weiss (1999)). When  $R$  is positive semidefinite, the problem is called *singular*, and *cheap* when  $R$  is zero. The singular and cheap cases have been treated within the framework of geometric control theory, see for example Hautus and Silverman (1983), Saberi and Sannuti (1987), Willems, Kîtapçı and Silverman (1986), and references therein. In particular, in Hautus and Silverman (1983) and Willems et al. (1986) it was proved that an optimal solution of the singular problem exists for all initial conditions if the class of allowable controls is extended

to include distributions. The approach taken in Saberi and Sannuti (1987) is based on an analysis of the structure of both singular and cheap problems (where the latter is treated as the limiting case of the former) carried out by means of the so-called *special coordinate basis*. Valuable results on this issue have also been obtained in Saberi, Sannuti and Chen (1995), Stoorvogel and Saberi (1998) and Stoorvogel (1992a,b), where the solution of the singular LQ problems is based on linear matrix inequalities.

This paper proposes a new perspective for the solution of the cheap LQ regulator. The keystone is the interpretation of the LQ regulator as an output nulling problem referred to the Hamiltonian system. This new approach enables to solve the cheap LQ regulator by only using the basic and computationally efficient tools of the geometric approach, namely the algorithms for controlled and conditioned invariance and invariant zeros, (Basile & Marro, 1969, 1992; Trentelman, Stoorvogel & Hautus, 2001; Wonham, 1985). By writing the conditions for optimality in the form of the Hamiltonian system, whose output has to be maintained identically equal to zero, the cheap LQ problem reduces to finding a state feedback such that the state-costate trajectory entirely lies on an internally stabilizable output nulling subspace of the Hamiltonian system. In the framework of the geometric approach, this is a standard unknown-input decoupling problem.

This work is intended to be the counterpart of Marro, Prattichizzo and Zattoni (2002) for the continuous time. The continuous time case is different from the discrete time one. In fact, differently from the discrete time, in the continuous time

<sup>☆</sup> This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Faryar Jabbari under the direction of Editor Roberto Tempo. Partially supported by the ministry of higher education of Italy (MIUR) under the project *Identification and Control of Industrial Systems*, by Fondazione Monte dei Paschi di Siena (2006–2008), by European Community (FP7-ICT-2007-215190) and by the Australian Research Council (DP0664789).

\* Corresponding author.

E-mail addresses: [prattichizzo@ing.unisi.it](mailto:prattichizzo@ing.unisi.it) (D. Prattichizzo), [ln@ee.unimelb.edu.au](mailto:ln@ee.unimelb.edu.au) (L. Ntogramatzidis), [gmarro@deis.unibo.it](mailto:gmarro@deis.unibo.it) (G. Marro).

the optimal state-feedback does not exist for any initial condition, so that significant issues related to existence of optimal solutions arise. Solving the cheap LQ problem by directly referring to the Hamiltonian system provides a very simple characterization of the subspace of all admissible initial conditions such that a state-feedback solution exists.

*Notation.* The symbol  $\mathbb{R}^{n \times m}$  is used to denote the space of  $n \times m$  real constant matrices. The image and the null-space of matrix  $M \in \mathbb{R}^{n \times m}$  are, denoted by  $\text{im } M$  and  $\ker M$  denoted by  $\text{im } M$  and  $\ker M$ , respectively. Given a subspace  $\mathcal{Y}$  of  $\mathbb{R}^n$ , the symbol  $M^{-1}\mathcal{Y}$  stands for the inverse image of  $\mathcal{Y}$  with respect to the linear transformation  $M$ , while  $\mathcal{Y}^\perp$  represents the orthogonal complement of  $\mathcal{Y}$ . Denote by  $M^T$  and by  $M^\dagger$  the transpose and the Moore–Penrose pseudo-inverse of  $M$ , respectively; since the identity  $(M^{-1})^T = (M^T)^{-1}$  holds, the symbol  $M^{-T}$  is used concisely. The symbol  $I_d$  will stand for the  $d \times d$  identity matrix. Finally, the symbol  $\mathbb{N}$  denotes the set of non-negative integer numbers.

## 2. Statement of the problem

Consider the linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), & x(0) &= x_0, \\ y(t) &= Cx(t), \end{aligned} \quad (1)$$

where, for all  $t \geq 0$ ,  $x(t) \in \mathbb{R}^n$  is the state,  $u(t) \in \mathbb{R}^m$  is the control input,  $y(t) \in \mathbb{R}^p$  is the output,  $A$ ,  $B$  and  $C$  are real constant matrices of proper sizes. With no loss of generality it is assumed that  $B$  has linearly independent columns and  $C$  has linearly independent rows. System (1) is briefly referred to as the triple  $(A, B, C)$ , and is concisely denoted by  $\Sigma$ . We make the following standing assumptions on  $\Sigma$ :

- (A1) the pair  $(A, B)$  is stabilizable;
- (A2)  $\Sigma$  has no invariant zeros on the imaginary axis.

The cheap LQ problem considered in this paper can be formulated as follows.

**Problem 2.1.** Find the subspace  $\mathcal{L}$  of initial conditions of  $\Sigma$  for which a matrix  $K \in \mathbb{R}^{m \times n}$  exists such that:

- (1) the closed-loop matrix  $A - BK$  is stable;
- (2) for all  $x_0 \in \mathcal{L}$ , the state trajectory generated by (1) with initial state  $x(0) = x_0$  and with the input  $u(t) = -Kx(t)$  minimizes the quadratic cost function

$$J = \frac{1}{2} \int_0^\infty y^\top(t) y(t) dt = \frac{1}{2} \int_0^\infty x^\top(t) C^\top C x(t) dt.$$

For  $x_0 \in \mathcal{L}$ , determine  $K$  such that (1)–(2) are satisfied.

Problem 2.1 is usually referred to as a *cheap* LQ problem, since the input function is not explicitly weighted in the performance index  $J$ . Differently from other approaches, here the optimal control is sought within the class of static state feedback inputs, so that optimal solutions containing distributions are ruled out. As will become clear in Section 4, this poses a problem of existence of optimal solutions to Problem 2.1. When Problem 2.1 admits solutions, the optimal feedback matrix does not depend on the particular initial state in  $\mathcal{L}$ . This issue is discussed in Section 5.

## 3. Geometric background

The geometric setting developed here requires the following notations:  $\mathcal{V}_\Sigma^*$  stands for the largest  $(A, \text{im } B)$ -controlled invariant subspace of  $(A, B, C)$  contained in  $\ker C$ , which is also denoted by  $\max \mathcal{V}(A, \text{im } B, \ker C)$ . Recall that  $\mathcal{V}_\Sigma^*$  is the subspace of all initial states  $x_0 \in \mathbb{R}^n$  of (1) for which an input exists such that the corresponding output is identically zero, and can be computed as the last term of the monotonically non-increasing sequence of subspaces  $(\mathcal{V}_i)_{i \in \mathbb{N}}$  described by

$$\mathcal{V}_0 = \ker C, \quad \mathcal{V}_i = \ker C \cap A^{-1}(\mathcal{V}_{i-1} + \text{im } B), \quad i = 1, \dots, k$$

where the integer  $k \leq n-1$  is determined by the condition  $\mathcal{V}_{i+1} = \mathcal{V}_i (= \mathcal{V}_\Sigma^*)$ , see e.g., Basile and Marro (1992, p. 210). The symbol  $\mathcal{S}_\Sigma^*$  stands for the smallest  $(A, \ker C)$ -conditioned invariant subspace containing  $\text{im } B$ , which is denoted by  $\min \mathcal{S}(A, \ker C, \text{im } B)$ , and can be computed as the last term of the monotonically non-decreasing sequence  $(\mathcal{S}_i)_{i \in \mathbb{N}}$  described by

$$\mathcal{S}_0 = \text{im } B, \quad \mathcal{S}_i = \text{im } B + A(\mathcal{S}_{i-1} \cap \ker C), \quad i = 1, \dots, k,$$

where now the integer  $k \leq n-1$  is determined by the condition  $\mathcal{S}_{i+1} = \mathcal{S}_i (= \mathcal{S}_\Sigma^*)$ , see Basile and Marro (1992, p. 209). The notion and related geometric properties of left and right-invertibility are now briefly recalled, see Basile and Marro (1992, p. 237) and Trentelman et al. (2001, p. 189) for details.

**Property 3.1.** The following statements are equivalent:

- (1)  $\Sigma$  is left-invertible;
- (2)  $\mathcal{V}_\Sigma^* \cap \mathcal{S}_\Sigma^* = \{0\}$ ;
- (3)  $B^{-1}\mathcal{V}_\Sigma^* = \{0\}$ .

Roughly speaking, if  $\Sigma$  is not left-invertible, the input function corresponding to a given response (obtained with zero initial condition) can only be determined modulo  $B^{-1}\mathcal{V}_\Sigma^* \subseteq \mathbb{R}^m$ , see Basile and Marro (1973, Theorem 3). The subspace  $B^{-1}\mathcal{V}_\Sigma^*$  is therefore called *input functional unobservability subspace*. Dually, the following property holds.

**Property 3.2.** The following statements are equivalent:

- (1)  $\Sigma$  is right-invertible;
- (2)  $\mathcal{V}_\Sigma^* + \mathcal{S}_\Sigma^* = \mathbb{R}^n$ ;
- (3)  $C\mathcal{S}_\Sigma^* = \mathbb{R}^p$ .

If  $\Sigma$  is not right-invertible, the output function can be imposed modulo any complement of the subspace  $C\mathcal{S}_\Sigma^* \subseteq \mathbb{R}^p$ , see Basile and Marro (1973, Theorem 4). Thus, the subspace  $C\mathcal{S}_\Sigma^*$  is called *output functional reachability subspace*.

Denote by  $\Sigma^\top$  the dual of system  $\Sigma$ , described by the triple  $(A^\top, C^\top, B^\top)$ .

**Property 3.3.** System  $\Sigma$  is left (right) invertible if and only if its dual  $\Sigma^\top$  is right (left) invertible.

The proof of the former follows from the well known identities  $\mathcal{V}_\Sigma^* = (\mathcal{S}_{\Sigma^\top}^*)^\perp$  and  $\mathcal{S}_\Sigma^* = (\mathcal{V}_{\Sigma^\top}^*)^\perp$ , see e.g., Basile and Marro (1992, p. 209). Recall that, given a friend  $F$  of  $\mathcal{V}_\Sigma^*$ , i.e., a matrix such that  $(A + BF)\mathcal{V}_\Sigma^* \subseteq \mathcal{V}_\Sigma^*$ , the eigenvalues of  $A + BF$  restricted to  $\mathcal{V}_\Sigma^*$  are split into two sets. The eigenvalues of  $A + BF$  which are restricted to  $\mathcal{V}_\Sigma^* \cap \mathcal{S}_\Sigma^*$  are all assignable by a suitable choice of  $F$ . The eigenvalues of  $A + BF$  restricted to the quotient  $\mathcal{V}_\Sigma^*/\mathcal{V}_\Sigma^* \cap \mathcal{S}_\Sigma^*$  are fixed for any friend of  $\mathcal{V}_\Sigma^*$ , and are called *invariant zeros* of  $\Sigma$ . If all the invariant zeros of  $\Sigma$  have strictly negative real part,  $\Sigma$  is said to be *minimum phase* and  $\mathcal{V}_\Sigma^*$  is said to be *internally stabilizable*. The subspace  $\mathcal{R}_\Sigma^* := \mathcal{V}_\Sigma^* \cap \mathcal{S}_\Sigma^*$  is often referred to as the *reachable subspace* on  $\mathcal{V}_\Sigma^*$ , see Morse (1973). By virtue of Property 3.1, if  $\Sigma$  is left-invertible, the set of invariant zeros reduces to the set of eigenvalues of  $A + BF$  restricted to  $\mathcal{V}_\Sigma^*$ . For

a detailed analysis of the internal and external eigenstructure of a controlled invariant subspace we refer to Basile and Marro (1992, pp. 217–222) and Trentelman et al. (2001, pp. 89–96).

Notice that the dynamical system whose evolution is described by (1) can be written backwards in time as

$$\begin{aligned} \dot{z}(t) &= -Az(t) - Bu(t), \\ y(t) &= Cz(t). \end{aligned}$$

The triple  $(-A, -B, C)$ , here denoted by  $\Sigma^{-1}$ , is therefore referred to as the time-reversed system associated with  $\Sigma$ . Note that

$$\mathcal{V}_{\Sigma^*} = \mathcal{V}_{\Sigma^{-1}} \quad \text{and} \quad \mathcal{S}_{\Sigma^*} = \mathcal{S}_{\Sigma^{-1}}, \tag{2}$$

since the definitions of  $\mathcal{V}_{\Sigma^*}$  and  $\mathcal{S}_{\Sigma^*}$  do not depend on the sign of the matrices  $A$  and  $B$ . Hence, the following property holds.

**Property 3.4.** System  $\Sigma$  is left (right) invertible if and only if  $\Sigma^{-1}$  is left (right) invertible.

#### 4. A geometric insight into the structure of the Hamiltonian system

Recall that the optimal state trajectory and control law for Problem 2.1 satisfy the following equations

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3}$$

$$\dot{\lambda}(t) = -C^T C x(t) - A^T \lambda(t), \tag{4}$$

$$B^T \lambda(t) = 0, \tag{5}$$

obtained by extending the state  $x(t)$  of system (1) with the costate function  $\lambda(t) \in \mathbb{R}^n$  ( $t \geq 0$ ). Eqs. (3)–(5) can be obtained from the computation of the derivatives of the Hamiltonian function with respect to  $x(t)$ ,  $\lambda(t)$  and  $u(t)$ , see e.g., Lewis and Syrmos (1995, pp. 131–133).

A fundamental observation for the approach taken here is that (3)–(5) can be written in the compact form:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \tag{6}$$

$$\hat{y}(t) = \begin{bmatrix} 0 & B^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = 0,$$

and that (6) can be regarded as a  $2n$  order system whose output is identically zero. The matrices in (6) are, respectively, denoted by the symbols  $\widehat{A}$ ,  $\widehat{B}$  and  $\widehat{C}$ , while the triple  $(\widehat{A}, \widehat{B}, \widehat{C})$  is denoted by  $\widehat{\Sigma}$ , and is referred to as the *Hamiltonian system*. From the structure of the Hamiltonian system, which along with the boundary equations represents a set of necessary and sufficient conditions for optimality, it is clear that Problem 2.1 can be reformulated as the problem of finding the control law maintaining  $\hat{y}(t) = 0$  for all  $t \geq 0$  for the assigned initial condition  $x(0) = x_0$ , and such that the corresponding state trajectory converges to the origin as  $t$  approaches infinity. Clearly, this aim can be achieved if and only if the initial condition  $x_0$  is such that an initial value of the costate  $\lambda_0 := \lambda(0)$  exists so that  $\begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix}$  belongs to an internally stabilizable  $(\widehat{A}, \text{im } \widehat{B})$ -controlled invariant contained in the null-space of  $\widehat{C}$ .

The following lemma provides an important characterization of the geometric structure of the Hamiltonian system in terms of left and right-invertibility. The further assumption of left-invertibility of  $\Sigma$  is technical in nature and will be removed in Section 5.1.

**Lemma 4.1.** If  $\Sigma$  is left-invertible, the Hamiltonian system  $\widehat{\Sigma}$  is both left and right-invertible.

**Proof.** First, note that  $\widehat{\Sigma}$  is the series connection of  $\Sigma$  and the time-reversed representation of the dual of  $\Sigma$ , henceforth denoted by the symbol  $\Sigma^{-T}$ , and representing the triple  $(-A^T, -C^T, B^T)$ . As a consequence of Properties 3.3 and 3.4, it follows that  $\Sigma^{-T}$  is right-invertible. The block-structure of the Hamiltonian system matches that described in Marro et al. (2002), even if the inner structures of  $\Sigma$  and  $\Sigma^{-T}$  in the continuous and discrete cases are different. Hence, the proof of the right-invertibility of  $\widehat{\Sigma}$  can be carried out by employing the same arguments used in Marro et al. (2002, Lemma 1), which are based on the fact that the input reachability subspace of  $\Sigma$  and the orthogonal complement of the input unobservability subspace of  $\Sigma^{-T}$  are equal.<sup>1</sup> Finally, a simple computation shows that the time-reversed representation of the Hamiltonian system  $\widehat{\Sigma}^{-1}$  and its dual  $\widehat{\Sigma}^T$  are linearly equivalent, i.e., the triple  $(-\widehat{A}, -\widehat{B}, \widehat{C})$  is obtained from the triple  $(\widehat{A}^T, \widehat{C}^T, \widehat{B}^T)$  via a change of coordinates in  $\mathbb{R}^{2n}$  given by the matrix  $T = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ . As a result,  $\widehat{\Sigma}^{-1}$  and  $\widehat{\Sigma}$  are left-invertible, since such is  $\widehat{\Sigma}^T$ . ■

In the following, the symbols  $\mathcal{V}_{\widehat{\Sigma}}^* := \max \mathcal{V}(\widehat{A}, \text{im } \widehat{B}, \ker \widehat{C})$  and  $\mathcal{S}_{\widehat{\Sigma}}^* := \min \mathcal{S}(\widehat{A}, \ker \widehat{C}, \text{im } \widehat{B})$  will be used, consistently with the notation previously introduced.

**Corollary 4.1.** Let  $\Sigma$  be left-invertible. The set of invariant zeros of the Hamiltonian system  $\widehat{\Sigma}$  is mirrored with respect to the imaginary axis, and is devoid of zeros on the imaginary axis.

**Proof.** First, let us show that the invariant zeros of  $\widehat{\Sigma}$  are in pairs  $(z, -z)$ . Let  $z$  be an invariant zero of  $\widehat{\Sigma}$ , i.e., it is an eigenvalue of  $(\widehat{A} + \widehat{B}\widehat{F})$  restricted to  $\mathcal{V}_{\widehat{\Sigma}}^*$ , where  $\widehat{F}$  is a friend of  $\mathcal{V}_{\widehat{\Sigma}}^*$ , since, owing to Lemma 4.1,  $\widehat{\Sigma}$  is left-invertible. It follows that  $z$  is an invariant zero of  $\widehat{\Sigma}^T$ , hence it is also an invariant zero of  $\widehat{\Sigma}^{-1}$ , since in the proof of Lemma 4.1 it was shown that  $\widehat{\Sigma}^T$  and  $\widehat{\Sigma}^{-1}$  are linearly equivalent up to a change of basis. However, since owing to (2)  $\mathcal{V}_{\widehat{\Sigma}}^* = \mathcal{V}_{\widehat{\Sigma}^{-1}}^*$  and the eigenvalues of  $(\widehat{A} + \widehat{B}\widehat{F})$  and those of  $(-\widehat{A} - \widehat{B}\widehat{F})$  are opposite, it is found that  $-z$  is an invariant zero of  $\widehat{\Sigma}$ . Now we show that  $\widehat{\Sigma}$  has no invariant zeros on the imaginary axis. Suppose by contradiction that  $z_0 = j\omega_0$  is a zero of  $\widehat{\Sigma}$ ; by definition of zero, it follows that for a suitable initial condition  $\begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix}$  of  $\widehat{\Sigma}$ , the input  $u(t) = e^{z_0 t} = e^{j\omega_0 t}$  gives rise to an identically zero output  $\hat{y}$ , which implies that the state trajectory is optimal for a cheap problem with  $x(0) = x_0$ , (Mangasarian, 1966). However, since  $z_0$  cannot be a zero of  $\Sigma$  in view of Assumption (A2), the same control  $u(t) = e^{z_0 t} = e^{j\omega_0 t}$  gives rise to a non-zero output of the system  $\Sigma$ . More precisely,  $y(t) = y_0 e^{j\omega_0 t}$  for a suitable  $y_0 \in \mathbb{R}^p$ , (Zhou, Doyle & Glover, 1996, Lemma 3.36). When  $\omega_0 = 0$ , this output is constant. When  $\omega_0 \neq 0$ , since  $-z_0$  is a zero of the Hamiltonian system as well in view of the first part of the proof, then an input  $u(t) = \sin(\omega_0 t)$  gives rise to the output  $y(t) = y_0 \sin(\omega_0 t + \varphi_0)$ . In both cases,  $\int_0^\infty y^T(t)y(t) dt$  is not finite. On the other hand, a finite cost can always be achieved in view of Assumption (A1), so that this solution would not be optimal. ■

**Corollary 4.2.** Let  $\Sigma$  be left-invertible. The internal eigenvalues of  $\mathcal{V}_{\widehat{\Sigma}}^*$  are the invariant zeros of the Hamiltonian system  $\widehat{\Sigma}$ . These are in pairs of the type  $(z, -z)$ . This set of zeros includes the invariant zeros of  $\Sigma$ .

**Proof.** The proof follows from Lemma 4.1, which ensures that  $\mathcal{V}_{\widehat{\Sigma}}^*$  has no arbitrarily assignable poles, and from Corollary 4.1. The fact that the set of zeros of  $\widehat{\Sigma}$  includes the invariant zeros of  $\Sigma$  comes from the fact that  $\widehat{\Sigma}$  is the series connection of  $\Sigma$  and  $\Sigma^{-T}$ . ■

<sup>1</sup> Recall that if  $M \in \mathbb{R}^{n \times m}$  and  $\mathcal{Y}$  subspace of  $\mathbb{R}^n$ , there holds  $M^T \mathcal{Y}^\perp = (M^{-1} \mathcal{Y})^\perp$ , (Basile & Marro, 1992, Property 3.1–3, p. 128).



**Proof.** Since  $\Sigma$  is supposed to be left-invertible, Lemma 4.1 ensures that  $\mathcal{V}_{\Sigma}^*$  has no assignable poles. Moreover, from Corollary 4.1 it follows that the poles of  $\mathcal{V}_{\Sigma}^*$ , which are the invariant zeros of  $\widehat{\Sigma}$ , are pairs of the type  $(z, -z)$ , hence all stable-antistable owing to Assumption (A2). This set of zeros includes the invariant zeros of  $\Sigma$ . Since  $\widehat{\Sigma}$  is left and right-invertible, the matrix  $T_1 := [T' \ T'']$ , where  $\text{im } T' = \mathcal{V}_{\Sigma}^*$  and  $\text{im } T'' = \mathcal{V}_{\Sigma}^{*c}$ , is a basis matrix of  $\mathbb{R}^{2n}$ . Since  $\mathcal{V}_{\Sigma}^*$  is an  $(\widehat{A}, \text{im } \widehat{B})$ -controlled invariant subspace, a matrix  $\widehat{F} \in \mathbb{R}^{m \times 2n}$  exists such that  $\mathcal{V}_{\Sigma}^*$  is an  $(\widehat{A} + \widehat{B}\widehat{F})$ -invariant. By performing the coordinate transformation defined by  $T_1$ , one obtains the following partitioned structure  $\widehat{A}_{F, T_1} := T_1^{-1}(\widehat{A} + \widehat{B}\widehat{F})T_1 = \begin{bmatrix} M_{SU} & \times \\ 0 & \times \end{bmatrix}$ , where the eigenvalues of  $M_{SU} \in \mathbb{R}^{2(n-r) \times 2(n-r)}$  are the poles of  $\mathcal{V}_{\Sigma}^*$ , all unassignable by virtue of the left-invertibility of  $\widehat{\Sigma}$ , are the strictly stable invariant zeros of the Hamiltonian system and their opposite. Since the subspace of the internal modes of  $\mathcal{V}_{\Sigma}^*$  and that of the antistable ones are disjoint and both  $(\widehat{A} + \widehat{B}\widehat{F})$ -invariant, a further basis transformation  $T_2$  in  $\mathbb{R}^{2n}$  can be performed so as to split the stable modes from the antistable ones. The matrix  $\widehat{A}_{F, T_2}$  that corresponds to  $\widehat{A}_{F, T_1}$  in this new basis has the following structure  $\widehat{A}_{F, T_2} := T_2^{-1}\widehat{A}_{F, T_1}T_2 = \begin{bmatrix} M_S & 0 & \times \\ 0 & M_U & \times \\ 0 & 0 & \times \end{bmatrix}$ , where the eigenvalues of  $M_S, M_U \in \mathbb{R}^{(n-r) \times (n-r)}$  are, respectively, the stable and antistable zeros of the Hamiltonian system  $\widehat{\Sigma}$ . Hence, the first  $n - r$  columns of the product  $T_1 T_2$  define a basis for a controlled invariant subspace which is internally stabilizable, whose poles are the eigenvalues of  $M_S$ . Thus, this subspace is  $\widehat{\mathcal{V}}_R$ . ■

**Lemma 5.1.** Let Assumptions (A1) and (A2) hold. Let  $\begin{bmatrix} V_X \\ V_A \end{bmatrix}$  be a basis matrix of  $\widehat{\mathcal{V}}_R$  with  $V_X, V_A \in \mathbb{R}^{n \times (n-r)}$ . Matrix  $V_X$  is full rank.

**Proof.** By contradiction assume that  $\ker V_X \neq \{0\}$  and let  $N_{V_X}$  be a basis matrix of  $\ker V_X$ , then  $\text{im} \begin{bmatrix} 0 \\ V_A N_{V_X} \end{bmatrix} \subseteq \widehat{\mathcal{V}}_R$ . From the controlled invariance of  $\widehat{\mathcal{V}}_R$ , one gets  $(-C^T C V_X - A^T V_A) \ker V_X \subseteq V_A \ker V_X$ , and consequently

$$-A^T V_A \ker V_X \subseteq V_A \ker V_X. \tag{9}$$

Since  $\widehat{\mathcal{V}}_R \subseteq \ker \widehat{C}$ , it ensues that  $V_A \ker V_X \subseteq \ker(B^T)$ , and from (9) the subspace  $V_A \ker V_X$  is contained in the unobservability subspace of  $\Sigma^{-T}$ , which is by definition the largest  $A^T$ -invariant subspace contained in the null-space of  $B^T$ . Moreover, modes of this invariant subspace are stable since  $\widehat{\mathcal{V}}_R$  is an internally stable  $(\widehat{A} + \widehat{B}\widehat{F})$ -invariant. Contradiction arises from Assumption (A1), which implies that the uncontrollability part of  $\Sigma$  is stable, or equivalently that the unobservability subspace of  $\Sigma^{-T}$  is antistable. ■

The result given in Lemma 5.1 is similar to the one found for discrete time systems in Marro et al. (2002). However, while in the discrete case the dimension of  $\widehat{\mathcal{V}}_R$  equals the dimension of the state space  $n$ , in the continuous case such dimension is  $n - r$ , so that here full rankness of  $V_X$  does not imply invertibility of  $V_X$ , since  $V_X$  is not square in general.

In the following theorem, it is shown how the solution of the output nulling problem for  $\widehat{\Sigma}$  enables the solvability condition on the initial state to be expressed in terms of the projection of the subspace  $\widehat{\mathcal{V}}_R$  on the state-space of  $\Sigma$ . In other words, when the initial state is such that an optimal solution exists, a basis for the subspace of admissible initial conditions can be used to directly compute the state feedback.

Differently from the discrete case, where the optimal gain involves the inverse of  $V_X$ , here as aforementioned  $V_X$  is not invertible, so that the Moore–Penrose pseudo-inverse has to be used instead of the inverse.

**Theorem 5.2.** Let  $\Sigma$  be left-invertible. Let Assumptions (A1) and (A2) hold. Problem 2.1 is solvable if and only if

$$x_0 \in \text{im } V_X \tag{10}$$

i.e., if and only if the initial condition  $x_0$  belongs to the projection of  $\widehat{\mathcal{V}}_R$  on the state-space  $\mathbb{R}^n$  whose dimension is  $n - r$ . If this is the case, let  $F_X, F_A \in \mathbb{R}^{m \times n}$  be obtained by partitioning a friend  $\widehat{F}_R$  of  $\widehat{\mathcal{V}}_R$  as  $\widehat{F}_R = [F_X \ F_A]$ . If condition (10) is met, the state-feedback matrix  $K$  solving Problem 2.1 is given by

$$K = -(F_X + F_A V_A V_X^\dagger). \tag{11}$$

**Proof.** Necessity and sufficiency of condition (10) can be easily proved by noting that (10) is equivalent to the existence of a costate initial condition  $\lambda_0 \in \mathbb{R}^n$  such that  $\begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix} \in \widehat{\mathcal{V}}_R$ . The dimension of  $\text{im } V_X$  is  $n - r$  in view of Lemma 5.1. Owing to Theorem 5.1, an extended feedback matrix  $\widehat{F}_R$  exists such that the extended trajectory obtained by applying the control  $u(t) = \widehat{F}_R \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$  entirely lies on  $\widehat{\mathcal{V}}_R$  and converges to the origin as  $t$  approaches infinity. Hence, the projection of the extended state on  $\mathbb{R}^n$  converges to zero as well. Hence, this is a feedback which steers the extended state along a stable trajectory evolving in  $\widehat{\mathcal{V}}_R$ . Since  $x(t) \in \text{im } V_X$ , it follows that the input function can be expressed as the state feedback  $u(t) = \widehat{F}_R \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = [F_X \ F_A] \begin{bmatrix} I_n \\ V_A V_X^\dagger \end{bmatrix} x(t)$ . Clearly, the dimension  $\text{im } V_X$  is  $n - r$  owing to Lemma 5.1. ■

As is clear from the proof of Theorem 5.2, the optimal static feedback  $K$  does not depend on the particular initial state satisfying (10). Moreover, under the assumption of left-invertibility of  $\Sigma$  the optimal control  $u(t) = -K x(t)$  is unique, due to the full rankness of  $V_X$ . In Section 5.1 it is shown that when the assumption of left-invertibility is removed, the optimal solution to Problem 2.1 is not unique.

We now show a simple formula for the optimal cost as a quadratic form of the initial state, which extends the very well-known result for the regular case, (Anderson & Moore, 1989; Lewis & Syrmos, 1995).

**Theorem 5.3.** Let  $x_0 \in \mathbb{R}^n$  be such that (10) is satisfied. The optimal value of the cost is  $J^* = x_0^T V_A V_X^\dagger x_0$ .

**Proof.** By using the Hamiltonian system (6) we find

$$\begin{aligned} J^* &= \int_0^\infty x^T(t) C^T C x(t) dt \\ &= \int_0^\infty (-x^T(t) \dot{\lambda}(t) - (\dot{x}^T(t) - u^T(t) B^T) \lambda(t)) dt \\ &= - \int_0^\infty \frac{d}{dt} (x^T(t) \lambda(t)) dt = x_0^T \lambda(0) = x_0^T V_A V_X^\dagger x_0. \quad \blacksquare \end{aligned}$$

### 5.1. Extension to non-left-invertible systems

In the case where system  $\Sigma$  is left-invertible, given an initial state  $x_0 \in \mathbb{R}^n$ , a solution to Problem 2.1 exists if and only if  $x_0$  satisfies (10), and the feedback solution  $u = -K x$ , with  $K$  defined in (11), is unique. When  $\Sigma$  is not left-invertible, the optimal feedback matrix is not unique. To see this, recall that if  $\Sigma$  is not left-invertible, the reachable subspace  $\mathcal{R}_\Sigma^*$  on  $\mathcal{V}_\Sigma^*$  differs from the origin. Recall that  $\mathcal{R}_\Sigma^*$  can be interpreted as the subspace of state trajectories that can be followed indefinitely while maintaining the output function  $y$  at zero. It follows that, if  $u^o$  is optimal for Problem 2.1 and  $x^o$  is the corresponding state trajectory, any other control leading to a state trajectory that differs from  $x^o$  only

by components on  $\mathcal{R}_\Sigma^*$  leads to the same value of the cost, and is therefore optimal as well. In other words, the set of optimal controls for Problem 2.1 is parameterized modulo the inputs driving the state of  $\Sigma$  on  $\mathcal{R}_\Sigma^*$ . Hence, when  $\Sigma$  is not left-invertible, the results of Theorems 5.1 and 5.2 do not directly apply. However, the geometric approach taken here can be easily adapted to handle non-left-invertible systems, see also Marro et al. (2002). Let  $F$  be such that  $(A + BF)\mathcal{V}_\Sigma^* \subseteq \mathcal{V}_\Sigma^*$ , and such that all the eigenvalues of  $(A + BF)$  restricted to  $\mathcal{R}_\Sigma^* = \mathcal{V}_\Sigma^* \cap \mathcal{J}_\Sigma^*$  (which are freely assignable) are stable; let also  $\Omega$  be a basis matrix of  $(B^{-1}\mathcal{V}_\Sigma^*)^\perp$ . Consider a non-left-invertible system  $\Sigma$ , and the auxiliary system  $\tilde{\Sigma}$  described by the triple  $(A + BF, B\Omega, C)$ . The system  $\tilde{\Sigma}$  is left-invertible as shown in Ntogramatzidis and Prattichizzo (2007, Theorem 1), since the introduction of matrix  $\Omega$  restricts the set of allowable inputs so as to eliminate the functions yielding motions on  $\mathcal{R}_\Sigma^*$ . Moreover, the invariant zeros of this new system  $\tilde{\Sigma}$  are those of the original system  $\Sigma$  plus those assigned through  $F$ , which are stable. Now, let  $\tilde{K}$  be the optimal state feedback matrix for the auxiliary system. Then  $K := \Omega\tilde{K} - F$  is one of the solutions of the original problem. All the other optimal solutions can be obtained by changing  $F$  within the set of stabilizing friends of  $\mathcal{V}_\Sigma^*$ . This set is always non-empty when  $\Sigma$  is non-left-invertible.

## 6. Concluding remarks

A new method for the solution of the cheap linear quadratic optimal control problem has been presented for continuous time systems. The key idea is to recast the cheap LQ regulator as an output nulling problem for the Hamiltonian system. The optimal control is sought within the class of input functions that can be expressed as a static state feedback, so as to exclude the possibility of distributions in the optimal control function. As a result of this, for certain initial conditions a solution of the cheap LQ problem may not exist. As such, a geometric analysis of independent theoretical interest on the structure and properties of the Hamiltonian system has been carried out in order to characterize the subspace of admissible initial conditions, and to determine the corresponding optimal state feedback matrix when an optimal solution does exist.

## References

- Anderson, B. D. O., & Moore, J. B. (1989). *Optimal control: Linear quadratic methods*. London: Prentice Hall Int.
- Basile, G., & Marro, G. (1969). Controlled and conditioned invariant subspaces in linear system theory. *Journal of Optimization Theory and Applications*, 3(5), 306–315.
- Basile, G., & Marro, G. (1973). A new characterization of some structural properties of linear systems: Unknown-input observability, invertibility and functional controllability. *International Journal of Control*, 17(5), 931–943.
- Basile, G., & Marro, G. (1992). *Controlled and conditioned invariants in linear system theory*. Englewood Cliffs, New Jersey: Prentice Hall, [On-line]. Available at <http://www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/downloads.htm>.
- Hautus, M. L. J., & Silverman, L. M. (1983). System structure and singular control. *Linear Algebra and Its Applications*, 50, 369–402.
- Ionescu, V., Oarã, C., & Weiss, M. (1999). *Generalized riccati theory and robust control, a popov function approach*. Wiley.
- Lewis, F. L., & Syrmos, V. (1995). *Optimal control*. New York: John Wiley & Sons.
- Mangasarian, O. L. (1966). Sufficient conditions for the optimal control of nonlinear systems. *SIAM Journal on Control*, 4(1), 139–152.
- Marro, G., Prattichizzo, D., & Zattoni, E. (2002). A geometric insight into the discrete time cheap and singular LQR problems. *IEEE Transactions on Automatic Control*, 47(1), 102–107.
- Morse, A. S. (1973). Structural invariants of linear multivariable systems. *SIAM Journal on Control*, 11(3), 446–465.
- Ntogramatzidis, L., & Prattichizzo, D. (2007). Squaring down LTI systems: A geometric approach. *Systems & Control Letters*, 56(3), 236–244.
- Saber, A., & Sannuti, P. (1987). Cheap and singular controls for linear quadratic regulators. *IEEE Transactions on Automatic Control*, AC-32(3), 208–219.
- Saber, A., Sannuti, P., & Chen, B. M. (1995). *H<sub>2</sub> optimal control. System and control engineering*. London: Prentice Hall International.
- Stoorvogel, A. A. (1992a). The singular  $H_2$  control problem. *Automatica*, 28(3), 627–631.
- Stoorvogel, A. A. (1992b). *The H<sub>∞</sub> control problem*. New York: Prentice Hall.
- Stoorvogel, A. A., & Saber, A. (1998). The discrete-time algebraic Riccati equation and linear matrix inequality. *Linear Algebra and its Applications*, 274, 317–365.
- Trentelman, H. L., Stoorvogel, A. A., & Hautus, M. (2001). *Control theory for linear systems. In Communications and control engineering*. Great Britain: Springer.
- Willems, J. C., Kitapçı, A., & Silverman, L. M. (1986). Singular optimal control: A geometric approach. *SIAM Journal of Control and Optimization*, 24(2), 323–337.
- Wonham, W. M. (1985). *Linear multivariable control: A geometric approach* (3rd ed.). New York: Springer-Verlag.
- Zhou, K., Doyle, J., & Glover, K. (1996). *Robust and optimal control*. New York: Prentice Hall.



**Domenico Prattichizzo** M.S. degree in Electronics Engineering and the Ph.D. degree in Robotics and Automation from the University of Pisa in 1991 and 1995, respectively. Since 2002 Associate Professor of Robotics at the University of Siena. In 1994, Visiting Scientist at the MIT AI Lab. Co-editor of two books by STAR, Springer Tracks in Advanced Robotics, Springer (2003, 2005). Since 2007 Associate Editor in Chief of the IEEE Trans. on Haptics. From 2003 to 2007, Associate Editor of the IEEE Trans on Robotics and Trans. on Control Systems Technologies. Member of the Editorial Board of many Conferences on Control and Robotics. Research interests are in geometric control and robotics. Author of more than 150 papers in the area of robotics and automatic control.



**Lorenzo Ntogramatzidis** received the “Laurea” degree, cum laude, in Computer Engineering in 2001 from the University of Bologna, Italy. He received the Ph.D. degree in Automatic Control and Operations Research in May 2005. Since August 2005, he has been with the Department of Electrical and Electronic Engineering, The University of Melbourne, Australia, where is currently a post-doctoral research fellow. His research interests lie in the broad area of systems and control theory.



**Giovanni Marro** graduated in Electrical Engineering at the University of Bologna, Italy, where he is Full Professor since 1970. He has also been lecturer of several courses in the field of system theory at the Universities of Bari and Parma. He has spent some periods at several universities in the United States, among them: Case Institute of Technology, Cleveland, Ohio, University of California, Berkeley, and University of Florida, Gainesville. He is co-author, together with his colleague G. Basile, of the well-known first paper of the geometric approach where controlled and conditioned invariants were introduced (JOTA, 1969). His research interests include the use of controlled invariants in control theory, dual lattice theory for approaching stabilizability, and optimization theory. His contributions are documented by about two hundreds of scientific papers and six books.