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A unified algorithmic setting for signal-decoupling compensators and unknown-input observers

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Abstract

A standard geometric-type environment, where only the very basic tools of the geometric approach are used (those supported by well-settled and well-tested computational aids) enables the development of algorithms for numerous control and estimation problems in the discrete-time case. These are: measurable or previewed signal localization problems, perfect or almost perfect tracking (right inversion), and, by duality, perfect or almost perfect unknown-input estimation with possible postknowledge and input reconstruction (left inversion). It is also shown that the devices obtained (compensator and observer), that may be noncausal when specific stability requirements are not met, can be implemented as dynamical systems including finite-horizon convolutors or finite impulse response (FIR) systems.

Keywords: discrete-time systems, disturbance decoupling, unknown-input observers, perfect tracking, geometric approach.

1 Introduction

This contribution considers two problems: perfect or almost perfect decoupling of a signal measurable or known in advance and perfect or almost perfect unknown-input observation of a linear function of the state with a possible postknowledge. They are the natural extensions of the classical disturbance decoupling and unknown-input observation problems, whose duality was pointed out in the very early approaches with geometric tools. Its motivation is the recent growth of importance and interest in the second problem, due to its connection with fault detection: a considerable number of new solutions to the unknown-input observation problem have recently been published, most of them giving numerical examples, that can also be easily handled and extended with the techniques described in this paper.

It is herein shown that a very complete and easily mechanizable algorithmic support is achievable for both problems in the discrete-time case, but also valid in particular cases for continuous-time systems. This is pointed out by solving a "benchmark example" often considered in the literature. Minimality of the decoupling compensators or unknown-input observers is obtained by using for their synthesis the minimal element of the lattice of self-bounded controlled invariants, see [1]. Only the former concepts and algorithms of the geometric approach (controlled and conditioned invariants) are used, without any need for their extensions (output nulling subspaces or almost controlled and conditioned invariants). The algorithms herein presented are implementable with the standard geometric approach software.

The main difference between the approach presented in this paper and previous investigations is that the possibility of trading stabilizability with preview (in the control problem) or postknowledge (in the observation problem) is pointed out for the first time with very simple and intuitive geometric-type arguments. In fact, only the relative-degree preaction or postknowledge were considered before.

The following notation is used. $\mathbf{R}$ stands for the field of real numbers. Sets, vector spaces and subspaces are denoted by script capitals like $\mathcal{V}$, matrices and linear maps by slanted capitals like A, the image and the null space of A by $\operatorname{im} A$ and $\ker A$, respectively, the transpose of A by A^{T}, the pseudoinverse by $A^{\#}$, and the spectrum by $\sigma(A)$.

2 The basic problems considered

In this section, the two basic problems, that are dual with each other, are stated, and a set of necessary and sufficient conditions for their solvability, corresponding to different (less or more restrictive) requirements on preaction or postknowledge, are derived.

2.1 Previewed signal decoupling

Consider the discrete time-invariant dynamical system Σ described by:

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) + H\,h(k)\,, \\ y(k) &= C\,x(k)\,, \end{aligned} \tag{1}$$

with state $x \in \mathbf{R}^n$, control input $u \in \mathbf{R}^p$, (possibly) previewed input $h \in \mathbf{R}^s$, controlled output $y \in \mathbf{R}^q$, matrices B and H full column rank and matrix C full row rank. The following standard symbols, referring to system (1), are used: $\mathcal{B}$ for $\operatorname{im} B$, $\mathcal{H}$ for $\operatorname{im} H$, $\mathcal{C}$ for $\ker C$, $\mathcal{V}^*$ for the maximum $(A, \mathcal{B})$-controlled invariant contained in

$\mathcal{C}$ [$\max \mathcal{V}(A,\mathcal{B},\mathcal{C})$], $\mathcal{V}_g^*$ for the maximum internally stabilizable $(A,\mathcal{B})$-controlled invariant contained in $\mathcal{C}$, $\mathcal{S}^*$ for the minimum $(A,\mathcal{C})$-conditioned invariant containing $\mathcal{B}$ [$\min \mathcal{S}(A,\mathcal{C},\mathcal{B})$] and $\mathcal{R}_{\mathcal{V}^*}$ for the reachable set on $\mathcal{V}^*$, computable as $\mathcal{V}^* \cap \mathcal{S}^*$. Also recall that (A,B,C) is left-invertible if $\mathcal{V}^* \cap \mathcal{S}^* = \{0\}$, and right-invertible if $\mathcal{V}^* + \mathcal{S}^* = \mathbf{R}^n$. Let us refer to the block diagram shown in Fig. 1, where a possible k_p-step preview of input h is represented with a delay before application to Σ, and Σ_c denotes the compensator to be derived.

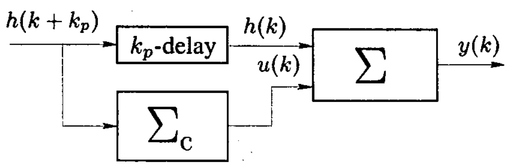

Figure 1: Decoupling of a measurable or previewed signal.

Problem 1 (decoupling of a measurable or previewed input) *Refer to system (1) with zero initial condition and assume that it is stable and that input h is previewed by k_p instants of time, $0 < k_p \leq \infty$. Determine a compensator having $h(k+k_p)$ as input and $u(k)$ as output, that sets $y(k)$ to be identically zero while maintaining the state of Σ bounded.*

Remark 1 (extension to non-purely dynamic systems) *If the second equation in (1) is generalized as $y(k) = C\,x(k) + D\,u(k) + M\,h(k)$, insert a dummy unit delay at the output and include it in the system equations, thus recovering a purely dynamic system for which Problem 1 has clearly the same solution.*

The following Property 1 is an extension of the classical disturbance decoupling problem ([2], [17]). It was presented in [16], while its dual (Property 3) had been presented just one year before in [4], all referring to continuous-time systems, thus requiring derivatives of input in control or of output in observation. Insights in meaning and proof are particularly straightforward in the discrete-time case. The algorithm presented in the next section constructively proves the "if" part, while the "only if" part is a consequence of the maximality of $\mathcal{V}^*$ as a locus of initial states in $\mathcal{C}$ corresponding to trajectories indefinitely controllable in $\mathcal{C}$, and to the maximality of $\mathcal{S}^*$ as a set of states that can be reached from the origin in a finite number of steps with all the intermediate states in $\mathcal{C}$ except the last one.

Property 1 (structural condition) *Problem 1 is solvable if and only if*

$$\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}^* . \tag{2}$$

The structural condition (2) guarantees that Problem 1 is solvable, possibly with $k_p = \infty$. If the more restrictive necessary and sufficient condition stated in the following Property 2 is satisfied, solution of Problem 1 is achievable with a finite value of k_p.

Property 2 (preview time and stability condition) *Refer to the standard conditioned invariant algorithm*

$$\begin{aligned} \mathcal{S}_0 &:= \mathcal{B}\,, \\ \mathcal{S}_i &:= A\,(\mathcal{S}_{i-1} \cap \mathcal{C}) + \mathcal{B}\,, \quad i = 1,\ldots,\rho_m\,, \end{aligned} \tag{3}$$

with $\mathcal{S}_{\rho_m} = \mathcal{S}^*$. Problem 1 is solvable with a finite preview time $k_p = \rho$ if and only if

1. $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}_\rho$;

2. $\mathcal{V}_m := \mathcal{V}^* \cap \min \mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{H})$ is internally stabilizable.

The condition stated in Property 2 is an improvement of the well-known condition $\mathcal{H} \subseteq \mathcal{V}_g^* + \mathcal{S}_\rho$ (where $\mathcal{V}_g^*$ denotes the restriction of $\mathcal{V}^*$ having only "good" modes, in Wonham's notation), first stated in [11]. It is more convenient from the algorithmic standpoint since it provides as $\mathcal{V}_m$ a minimal-dimension self-bounded controlled invariant, thus implying order reduction of the derived decoupling compensators or unknown-input observers in the dual case.

A proof of Property 2 is available in [1], where a different, equivalent expression for $\mathcal{V}_m$ is also provided.

Properties 1 and 2 have the following consequences.

1. Let us assume $\mathcal{H} = \mathbf{R}^n$ (all the components of the state are independently affected by h). By condition (2), in this case the problem is solvable if and only if $\mathcal{V}^* + \mathcal{S}^* = \mathbf{R}^n$, i.e., if and only if the triple (A,B,C) is right-invertible.
2. If conditions 1 and 2 in Property 2 are satisfied for $\rho = 0$, Problem 1 reduces to the standard localization or decoupling of a measurable disturbance. In this case the algorithmic setting considered in the next section can also be used for continuous-time systems without any need for differentiators.
3. Since the internal unassignable eigenvalues of $\mathcal{V}_m$ are a part of those of $\mathcal{V}^*$ – the invariant zeros of the triple (A,B,C) – condition 2 in Property 2 is satisfied if (but not only if) this triple is minimum-phase.
4. If condition 2 of Property 2 is not satisfied for $\rho = \rho_m$ (i.e., $\mathcal{V}_m$ is not internally stabilizable), an infinite or very large preview time is necessary and the compensator must include a FIR system. This is achievable in practice in some cases (for instance, in completely pre-programmed machine-tool or robot motion control ending in a rest position). However, almost perfect decoupling is possible with a FIR system when the preview time is significantly greater than the time constant of the most significant unstable internal unassignable eigenvalue of $\mathcal{V}_m$.

Let us now consider the dual setting.

2.2 Unknown-input estimation of inaccessible output or state

Refer to the discrete time-invariant dynamical system Σ_d described by:

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k)\,, \\ y(k) &= C\,x(k)\,, \\ e(k) &= E\,x(k)\,, \end{aligned} \qquad (4)$$

with state $x \in \mathbf{R}^n$, inaccesible input $u \in \mathbf{R}^p$, informative output $y \in \mathbf{R}^q$, output to be estimated $e \in \mathbf{R}^s$, matrix B full column rank and matrices C and E full row rank. The symbols $\mathcal{B}, \mathcal{C}, \mathcal{V}^*, \mathcal{S}^*$ denote the same subspaces already defined when dealing with Problem 1, while $\mathcal{E}$ is used for ker E. Let us refer to the block diagram shown in Fig. 2, where a possible k_p-step postknowledge of output e is represented with a delay and Σ_e denotes the estimator to be derived.

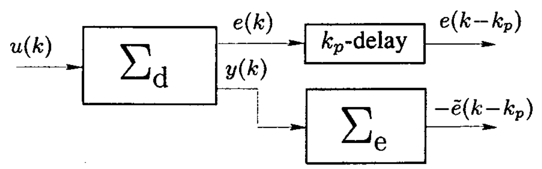

Figure 2: Unknown-input current or delayed observation of a linear function of the state.

Problem 2 (unknown-input current or delayed estimation of a linear function of the state) *Refer to system (4). Determine a stable dynamic observer, having $y(k)$ as input and giving as output $-\tilde{e}(k-k_p)$, an estimate of the opposite of the output $e(k)$ delayed by k_p instants of time, $0 < k_p \leq \infty$. In other words, the observer is required to maintain the estimation error $\eta(k-k_p) := e(k-k_p) - \tilde{e}(k-k_p)$ identically zero.*

Remark 2 (extension to non-purely dynamic systems) *If the second and third equation in (4) are generalized as $y(k) = C\,x(k) + D\,u(k)$, $e(k) = E\,x(k) + N\,u(k)$, insert a dummy unit delay at the input and include it in the system equations, thus recovering a purely dynamic system for which Problem 2 has clearly the same solution.*

Property 3 (structural condition) *Problem 2 is solvable if and only if*

$$\mathcal{E} \supseteq \mathcal{V}^* \cap \mathcal{S}^*\,. \qquad (5)$$

Note that condition (5) is the dual of (2). In fact, Problem 2 is the dual of Problem 1 (see [2], [12] and [5]), and all the previously stated conditions are still valid for it, provided that A is replaced by A^{T}, B by C^{T}, H by E^{T} and C by B^{T}. The preview time k_p becomes the postknowledge time in the dual problem. Duality of conditions (5) and (2) follows from the well-known identity

$$(\max \mathcal{V}(A, \operatorname{im} B, \ker C))^{\perp} = \min \mathcal{S}(A^{\mathrm{T}}, \ker B^{\mathrm{T}}, \operatorname{im} C^{\mathrm{T}})\,.$$

Under the above replacements of matrices, Property 2 is also valid for Problem 2, and the observer solving Problem 2 can be derived as the dual of the compensator solving Problem 1. The remarks that follows Property 2 can also be easily dualized. For instance, we conclude that, if $\mathcal{E} = \{0\}$ (i.e., a complete state estimation is required), Problem 2 is solvable if and only if $\mathcal{V}^* \cap \mathcal{S}^* = \{0\}$, i.e., if and only if the triple (A, B, C) is left-invertible. Infinite-time postknowledge also has a meaning, corresponding to the case where recorded data are processed in batch mode. However, almost perfect observation is also possible in this case with a FIR system if the assumed postknowledge time is significantly greater than the time constant of the most significant unstable internal unassignable eigenvalue of $\mathcal{V}_m$.

Remark 3 (right and left inversion) *Perfect tracking (right inversion) and reconstruction of an inaccessible input (left inversion) can be solved as particular cases of Problems 1 and 2 as shown in Fig. 3. Note that the extensions to non-purely dynamic systems considered in Remarks 1 and 2 are necessary in these cases.*

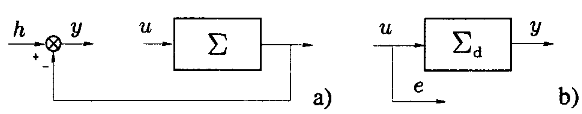

Figure 3: Schemes for multivariable right and left inversion.

3 An algorithmic solution

We consider here an algorithmic solution[1] of Problem 1. The basic idea, from a theoretical viewpoint, is synthesized as follows. Let us recall that $\mathcal{V}_m$ is a locus of initial states in $\mathcal{C}$ corresponding to trajectories indefinitely controllable in $\mathcal{C}$, while $\mathcal{S}_\rho$ is the maximum set of states that can be reached from the origin in $\rho+1$ steps with all the states in $\mathcal{C}$ except the last one. Suppose that an impulse is applied at input h at the time instant $\rho+1$, producing an initial state $x_h \in \mathcal{H}$, decomposable as $x_h = x_{h,s} + x_{h,v}$, with $x_{h,s} \in \mathcal{S}_\rho$ and $x_{h,v} \in \mathcal{V}_m$. Let us apply the control sequence that drives the state from the origin to $-x_{h,s}$ along a trajectory in $\mathcal{S}_\rho$, thus nulling the first component. The second component can be maintained on $\mathcal{V}_m$ by a suitable control action in the time interval $\rho+1 \leq k < \infty$ while avoiding divergence of the state if all the internal unassignable modes of $\mathcal{V}_m$ are stable or stabilizable. If not, it can be further decomposed as $x_{h,v} = x'_{h,v} + x''_{h,v}$, with $x'_{h,v}$ belonging to the subspace of the stable or stabilizable internal modes of $\mathcal{V}_m$ and $x''_{h,v}$ to that of the unstable modes. The former component can be maintained on $\mathcal{V}_m$ as before, while the latter can be nulled by reaching $-x''_{h,v}$ with a control action in the time interval $-\infty < k \leq \rho$ corresponding to a trajectory in $\mathcal{V}_m$ from the origin.

[1]Condition 1 in Property 2, implying the structural condition of Property 1, is assumed to be satisfied. Two different strategies are outlined according to whether condition 2 in Property 2 is satisfied or not.

The following algorithmic procedure yields a feasible feedforward compensator based on the above ideas, hence also supporting the non-minimum phase case. The algorithms herein presented require that system (1) is left-invertible with respect to the control input. Nevertheless, Algorithm 3 (listed next) provides a means to deal with non left-invertible systems. Algorithms 1 and 2 provide the control and state sequences for motions on $\mathcal{S}_\rho$ and $\mathcal{V}_m$, respectively, assuming $h(\cdot)=I\,\delta(k-\rho-1)$. To this aim, they require the decomposition of H as $H=VH_1'+SH_2'$, where V and S denote basis matrices of $\mathcal{V}_m$ and $\mathcal{S}_\rho$, respectively. Let us choose F such that $(A+BF)\,\mathcal{V}_m\subseteq\mathcal{V}_m$ and perform the state space basis transformation $T:=[\,V\;S\;T_1\,]$, where T_1 is such that T is nonsingular. The system matrices in the new basis have the structures

$$
\begin{aligned}
&A'=\begin{bmatrix} A'_{11} & A'_{12} & A'_{13}\\ 0 & A'_{22} & A'_{23}\\ 0 & A'_{32} & A'_{33}\end{bmatrix},\quad B'=\begin{bmatrix}0\\ B'_2\\ 0\end{bmatrix},\\
&C'=[\,0\;\; C'_2\;\; C'_3\,],\qquad H'=\begin{bmatrix}H'_1\\ H'_2\\ 0\end{bmatrix}.\\
&F'=[\,F'_1\;\; F'_2\;\; F'_3\,],
\end{aligned}
\tag{6}
$$

Algorithm 1 (Motion on $\mathcal{S}_\rho$) *The controls $U_1(k)$, $k=0,\ldots,\rho$, and the corresponding states $X_1(k)$, $k=1,\ldots,\rho+1$, are derived as follows.*

1. Compute basis matrices M_i for $\mathcal{S}_{i-1}\cap\mathcal{C}$, $i=1,\ldots,\rho$. Set $M_{\rho+1}:=S$ and $\beta(\rho+1):=H_2'$.

2. Compute the sequences $\beta(i)$ and $U_1(i)$, $i=1,\ldots,\rho$, as

$$
\begin{bmatrix}\beta(\rho-j)\\ U_1(\rho-j)\end{bmatrix}=
[\,A M_{\rho-j}\;\; B\,]^{\#}\,M_{\rho-j+1}\beta(\rho-j+1),\quad j=0,\ldots,\rho-1.
$$

3. Compute $U_1(0)$, driving the states from the origin to $M_1\beta(1)$, by $U_1(0)=B^{\#}M_1\,\beta(1)$.

4. Compute the intermediate states as $X_1(i)=M_i\,\beta(i)$, $i=1,\ldots,\rho+1$.

Algorithm 2 (Motion on $\mathcal{V}_m$) *If $\mathcal{V}_m$ is internally stabilizable the motion on $\mathcal{V}_m$ is provided by the pair (A'_{11},H'_1) in (6), i.e. the controls are $U_2(\rho+1+i)=F_1A_{11}^{i}\bar{H}_2$, $i=0,1,\ldots$, and the states $X_2(\rho+1+i)=A_{11}^{i}\bar{H}_2$, $i=0,1,\ldots$. Conversely, if $\mathcal{V}_m$ is not internally stabilizable, a further state space basis transformation T', whose aim is to separate the stable and unstable modes of $\mathcal{V}_m$, is required. Refer to the matrices A'_{11} and F'_1: the corresponding matrices A''_{11}, H'_1 and F''_1 in the new basis have the structures*

$$
A''_{11}=\begin{bmatrix}A_s & 0\\ 0 & A_u\end{bmatrix},\quad H''_1=\begin{bmatrix}H_s\\ H_u\end{bmatrix},\qquad F''_1=[\,F_s\;\;F_u\,].
$$

A preaction, nulling the unstable component of the state H_u at the time instant $\rho+1$, must be computed backward through the matrix A_u. The controls $U_3(\rho-j):=-F_uA_u^{-j-1}\bar{H}_u$, for $j=0,1,\ldots$, the states are $X_3(\rho-j):=-A_u^{-j-1}\bar{H}_u$, for $j=0,1,\ldots$. The stable component of the state H_s is managed as in case of stabilizable $\mathcal{V}_m$.

3.1 Design of the compensator

Let us refer to Fig. 4, which shows a typical input sequence achieving decoupling of an impulse at input h and time ρ in the most general case. If the internal modes of $\mathcal{V}_m$ (that are a subset of the invariant zeros of the plant from u to y) are all stable, decoupling is achieved by a *relative-degree dead-beat* (motion on $\mathcal{S}_\rho$), followed by a *postaction* (motion on $\mathcal{V}_m$ along these zeros) that can be realized with a stable purely dynamic system. In this case the compensator consists of a ρ-step FIR system and a dynamic unit, described by

$$
\begin{aligned}
u(k)&=\sum_{\ell=0}^{\rho}\Phi(\ell)\,h(k-\ell)\,, &(7)\\
x(k)&=\sum_{\ell=1}^{\rho+1}\Psi(\ell)\,h(k-\ell)\,, &(8)\\
w(k+1)&=N\,w(k)+L\,h(k-\rho-1)\,, &(9)\\
u(k)&=M\,w(k)\,, &(10)\\
x(k)&=V\,w(k)\,, &(11)
\end{aligned}
$$

for $k=0,1,\ldots$, with $\Phi(i):=U_1(i)$ $(i=0,\ldots,\rho)$, $\Psi(i):=X_1(i)$ $(i=1,\ldots,\rho+1)$, $N:=A'_{11}$, $L:=H'_1$, $M:=F'_1$. If, on the other hand, unstable modes are present in $\mathcal{V}_m$, a *preaction* is also required. Since the evolution of the state according to the unstable modes of $\mathcal{V}_m$ can only be computed backward and reproduced through a FIR system only, the time interval characterizing the FIR is enlarged to include the preaction time k_a, large enough to make the decoupling error negligible. Thus, eqs. (7) and (8) are modified as

$$
\begin{aligned}
u(k)&=\sum_{\ell=-k_a}^{\rho}\Phi(\ell)\,h(k-\ell)\,, &(12)\\
x(k)&=\sum_{\ell=-k_a+1}^{\rho+1}\Psi(\ell)\,h(k-\ell)\,, &(13)
\end{aligned}
$$

with $\Phi(i):=U_1(i)+U_3(i)$ $(i=-k_a,\ldots,\rho)$ and $\Psi(i):=X_1(i)+X_3(i)$ $(i=-k_a,\ldots,\rho+1)$. The dynamic unit is still described by (9), (10) and (11) with $N:=A_s$, $L:=H_s$, $M:=F_s$.

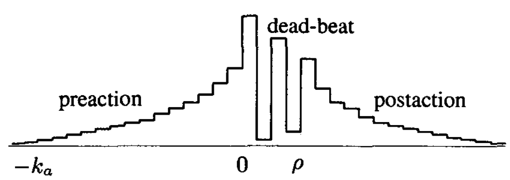

Figure 4: An input sequence for decoupling at time ρ.

If the triple (A,B,C) is not left-invertible, the previous procedure can be applied anyhow, provided that a preliminary manipulation is performed to obtain a left-invertible triple and the results obtained are suitably adapted to fit the original system. The following algorithm traces the way for this.

Algorithm 3 (Extension to non left-invertible systems) *If the triple (A,B,C) is not left-invertible, the previous procedure should be applied to (A^*,B^*,C), with*

1. $A^*:=A+BF^*$, where F^* is a state feedback matrix such that $(A+BF^*)\mathcal{V}^*\subseteq\mathcal{V}^*$ and all the elements of $\sigma(A+BF^*)|_{\mathcal{R}_{\mathcal{V}^*}}$ are stable;

2. $B^*:=BU^*$, where U^* is a basis matrix of the subspace $\mathcal{U}^*:=(B^{-1}\mathcal{V}^*)^{\perp}$, the orthogonal complement of the inverse image of $\mathcal{V}^*$ with respect to B.

Let $\bar{U}_i(k)$ and $\bar{X}_i(k)$ ($i=1,2,3$ and k consistently defined) be the sequences of controls and states provided by Algorithms 1 and 2 applied to (A^*,B^*,C). The corresponding control sequences for (A,B,C) are to be computed as $U_i(k)=U^*\bar{U}_i(k)+F^*\bar{X}_i(k)$ ($i=1,2,3$).

4 Illustrative examples

4.1 Example 1

Let us consider the signal decoupling problem for system (1) with

$$A=\begin{bmatrix} .5 & 0 & 0 \\ -1 & .2 & .5 \\ 0 & .3 & .4 \end{bmatrix},\ B=\begin{bmatrix} -1 & 0 \\ 1 & -2 \\ 2 & 1 \end{bmatrix},$$
$$H=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ C=\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{14}$$

Standard computations[2] provide

$$\mathcal{V}^*=\text{im}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$
$$\mathcal{S}^*=\mathcal{B},\quad \mathcal{S}_0=\mathcal{S}^*,\quad \mathcal{V}_m=\mathcal{V}^*. \tag{15}$$

It follows that $\rho=0$. The system is left-invertible and the unique internal eigenvalue of $\mathcal{V}_m$ is $z=.8$. Hence, in this case it is possible to derive a single-step FIR system and a stable postaction dynamic compensator with a single state, described by the equations (7), (8), (9), (10) and (11) with

$$\Phi(0)=\begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix},\quad N=[\ .8\ ],$$
$$L=[\ -5\ \ 1\ \ 2\ ],\quad M=\begin{bmatrix} 0 \\ -.3 \end{bmatrix}. \tag{16}$$

This solution applies both for the output decoupling of a measurable disturbance affecting all the state components and the unknown-input complete state observation of the dual of system (14), that is achieved with the dual of (16). In these cases no preaction or postknowledge is required, and the obtained compensator or observer are strictly stable non-purely dynamic systems.

4.2 Example 2

Consider again the matrices in (14), but with $B_{22}=-3.5$. The subspaces (15) are the same, but the internal eigenvalue of $\mathcal{V}_m$ is $z=1.25$, hence unstable, so that the decoupling is obtainable only with preaction (and the input to be decoupled must be known in advance) or the unknown-input complete state observation only with postknowledge. The synthesis procedure in this case yields

$$U_1(0)=\begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix},$$
$$U_3(-j)=-F_u\,A_u^{-j-1}\,\bar{H}_u,\ \ j=0,1,\dots,k_a, \tag{17}$$

with

$$A_u=[\ 1.25\ ],\quad \bar{H}_u=[\ -8\ \ 1\ \ 3.5\ ],$$
$$F_u=\begin{bmatrix} 0 \\ -.3 \end{bmatrix}, \tag{18}$$

describing a FIR system with $\Phi(i)=U_1(i)+U_3(i)$ ($i=-k_a,\dots,0$) (with $U_1(i)$ assumed to be zero if not explicitly defined) that approximates an antistable non-purely dynamic system. Postaction is not required in this case. With $k_a=30$ the decoupling error for a unit impulse at input h is about 10^{-3}, while with $k_a=60$ it reduces to about 10^{-6}. The same results hold in the dual case for the error in the delayed observation of the state in the presence of a unit impulse applied at the unknown input.

4.3 Example 3

Let us now consider a standard example known in the literature [15], [9], [18], concerning the complete state observation (hence with $E=I$) of the continuous-time triple (A,B,C) defined by

$$A=\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix},\quad B=\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},$$
$$C=\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

We refer to the dual problem, namely the output decoupling of a disturbance affecting the whole state (hence with $H_d:=I$) for the system $A_d:=A^{\mathrm{T}}$, $B_d:=C^{\mathrm{T}}$, $C_d:=B^{\mathrm{T}}$. The first computation concerns $\mathcal{V}^*$ and $\mathcal{S}^*$, that are

$$\mathcal{V}^*=\text{im}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\quad \mathcal{S}^*=\mathbf{R}^3.$$

The system is not left-invertible. Left-invertibility is achieved by using Algorithm 3, with the free eigenvalues

[2] These can be done with the Geometric Approach Toolbox for Matlab, first published with [3] and now available for free download on the web site http://www.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm.

set to -3 and -6, thus obtaining

$$A_d^* = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 18 & -9 \end{bmatrix}, \quad B_d^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$F_d^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 18 & -8 \end{bmatrix}.$$

Subsequent computations give $\mathcal{V}_m = \mathcal{V}^*$, $\rho = 0$, hence $\mathcal{S}_\rho = \mathcal{B}$, and the compensator provided by the algorithm in the previous section is

$$\Phi_d(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N_d = \begin{bmatrix} 0 & -1 \\ 18 & -9 \end{bmatrix},$$
$$L_d = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_d = \begin{bmatrix} 1 & 0 \\ 18 & -8 \end{bmatrix}.$$

By dualizing this compensator we obtain the unknown-input observer, that is of order two with arbitrary eigenvalues. This fits the results of the above-mentioned previous investigations. Note that the arbitrariness of the eigenvalues is related to lack of left invertibility.

5 Conclusion

A systematic algorithmic procedure has been presented for solving a class of feedforward control and observation problems by using the standard tools of the geometric approach. The main feature of the approach is that the minimum set of fixed poles of the compensator or observer is sharply selected, thus satisfactorily solving the stability problem.

Recently it has been pointed out that general H_2 optimal control problems can be solved as extensions of some standard problems of the geometric approach (e.g., disturbance localization with state or measurement feedback, see for instance [6]). An interesting feature of the theory presented in this paper is that it may open the way to the treatment of both preview optimal control and H_2 optimal control in a unified mathematical framework, where dynamic systems and FIR systems are considered together.

References

[1] F. Barbagli, G. Marro and D. Prattichizzo, "Solving signal decoupling problems through self-bounded controlled invariants", *39th CDC*, Sydney, Australia, 2000.

[2] G. Basile and G. Marro, "L'invarianza rispetto ai disturbi studiata nello spazio degli stati" ("Disturbance decoupling considered in the state space"), *Rendiconti della LXX Riunione Annuale AEI*, paper 1-4-01, Rimini, Italy, 1969.

[3] G. Basile and G. Marro, *Controlled and Conditioned Invariants in Linear System Theory*, Prentice Hall, Englewood Cliffs, New Jersey, 1992.

[4] G. Basile, F. Hamano, and G. Marro, "Some new results on unknown-input observability", *Proceedings of the 8th IFAC Congress*, paper no. 2.1, Kyoto, Japan,1981.

[5] G. Basile, G. Marro and A. Piazzi, "A new solution to the disturbance localization problem with stability and its dual", *Proceedings of the '84 International AMSE Conference on Modelling and Simulation*, vol. 1.2, pp. 19–27, Athens, 1984.

[6] A. Saberi, P. Sannuti and B.M. Chen, *H_2 Optimal Control*, Prentice Hall, Englewood Cliffs, New Jersey, 1995.

[7] M.B. Estrada and M. Malabre, "Necessary and sufficient conditions for disturbance decoupling with stability using PID control laws", *IEEE Transactions on Automatic Control*, vol. 44, no. 6, pp. 1311–1315, 1999.

[8] E. Gross, M. Tomizuka and W. Messner, "Cancellation of discrete time unstable zeros by feedforward control", *ASME Journal of Dynamic Systems, Measurement and Control*, vol. 116, no. 1, pp. 33–38, 1994.

[9] M. Hou and P.C. Müller, "Design of observers for linear systems with unknown inputs", *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 871–874, 1992.

[10] L.R. Hunt, G. Meyer and R. Su, "Noncausal inverses for linear systems", *IEEE Transactions on Automatic Control*, vol. 41, no. 4, pp. 608–611, 1996.

[11] H. Imai, M. Shinozuka, T. Yamaki, D. Li and M. Kuwana, "Disturbance decoupling by feedforward and preview control", *ASME Journal of Dynamic Systems, Measurements and Control*, vol. 105, no. 3, pp. 11–17, 1983.

[12] R. Laschi and G. Marro, "Alcune osservazioni sull'osservabilità dei sistemi dinamici con ingressi inaccessibili", ("Some remarks on observability of dynamic system with inaccessible inputs"), *Rendiconti della LXX Riunione Annuale AEI*, paper 1-1-06, Rimini, Italy, 1969.

[13] M. Massoumnia, "A geometric approach to the synthesis of failure detection filters", *IEEE Transactions on Automatic Control*, vol. 31, no. 9, pp. 839–846, 1986.

[14] J.M. Schumacher, "On a conjecture of Basile and Marro", *Journal of Optimization Theory and Applications*, vol. 41, n. 2, pp. 371–376, 1983.

[15] S.H. Wang, E.J. Davison and P. Dorato, "Observing the state of systems with unmeasurable disturbances", *IEEE Transaction on Automatic Control*, vol. AC-20, pp. 716–717, 1975.

[16] J.C. Willems, "Feedforward control, PID control laws, and almost invariant subspaces", *System & Control Letters*, vol. 1, no. 4, pp. 277–282, 1982.

[17] W.M. Wonham and A.S. Morse, "Decoupling and pole assignment in linear multivariable systems: a geometric approach", *SIAM Journal of Control and Optimization*, vol. 8, no. 1, pp. 1–18, 1970.

[18] F. Yang and R.W. Wilde, " Observers for linear systems with unknown inputs", *IEEE Transactions on Automatic Control*, vol. 33, no. 7, pp. 677–681, July 1988.