Substituting the previous matrices into (7.1), we get

$$\begin{aligned}\sum_{i=0}^{\omega}\left[J_j^i \otimes \begin{bmatrix}N_i\\ D_i\end{bmatrix}\right] &= \sum_{i=0}^{\omega}\left(s_j^i I_{p_j} + \cdots + s_j^0 C_i^i E_j^i\right) \otimes \begin{bmatrix}N_i\\ D_i\end{bmatrix}\\ &= \sum_{i=0}^{\omega}\left[s_j^i I_{p_j} C_i^0 \otimes \begin{bmatrix}N_i\\ D_i\end{bmatrix}\right]\\ &\quad + \sum_{i=0}^{\omega-1}\left[s_j^i E_j^1 C_{i+1}^1 \otimes \begin{bmatrix}N_{i+1}\\ D_{i+1}\end{bmatrix}\right] + \cdots\\ &\quad + \sum_{i=0}^{1}\left[s_j^i E_j^{\omega-1} C_{i+\omega-1}^{\omega-1} \otimes \begin{bmatrix}N_{i+\omega-1}\\ D_{i+\omega-1}\end{bmatrix}\right]\\ &\quad + \sum_{i=0}^{0}\left[s_j^i E_j^{\omega} C_{i+\omega}^{\omega} \otimes \begin{bmatrix}N_{i+\omega}\\ D_{i+\omega}\end{bmatrix}\right].\end{aligned}$$

Denote

$$\theta_j^{\omega-k} = \sum_{i=0}^{k}\left[s_j^i C_{i+\omega-k}^{\omega-k}\begin{bmatrix}N_{i+\omega-k}\\ D_{i+\omega-k}\end{bmatrix}\right], \qquad k = 0,\ldots,\omega. \tag{7.4}$$

Comparing (2.11) with (7.4), we clearly have

$$\theta_j^0 = \begin{bmatrix}N(s_j)\\ D(s_j)\end{bmatrix}. \tag{7.5}$$

So (7.1) can be simplified as

$$\begin{aligned}\sum_{i=0}^{\omega}\left[J_j^i \otimes \begin{bmatrix}N_i\\ D_i\end{bmatrix}\right] &= I_{p_j} \otimes \theta_j^0 + \cdots + E_j^{\omega} \otimes \theta_j^{\omega}\\ &= \begin{bmatrix}\theta_j^0 & \theta_j^1 & \cdots & \theta_j^{\omega} & 0 & \cdots & 0\\ & \theta_j^0 & \theta_j^1 & \ddots & \ddots & \ddots & \vdots\\ & & \theta_j^0 & \ddots & \ddots & \ddots & 0\\ & & & \ddots & \ddots & \ddots & \theta_j^{\omega}\\ & & & & \theta_j^0 & \theta_j^1 & \vdots\\ & & & & & \theta_j^0 & \theta_j^1\\ & & & & & & \theta_j^0\end{bmatrix}.\end{aligned}$$

Since the previous matrix has full-column rank if and only if $\theta_j^0, j = 1, 2, \ldots, w$, has full-column rank, i.e., the relation (2.2) holds, the conclusion of the lemma clearly holds true.

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A Unified Setting for Decoupling With Preview and Fixed-Lag Smoothing in the Geometric Context

Giovanni Marro, Domenico Prattichizzo, and Elena Zattoni

***Abstract*—Exact decoupling with preview, perfect tracking of previewed references, unknown-input state observation with fixed lag, and left inversion with fixed lag are considered from a unifying perspective where exact decoupling with preview is the basic problem. Necessary and sufficient constructive conditions for decoupling with finite preview are proved in the geometric framework. Structural and stabilizability conditions are considered separately and the use of self-bounded controlled invariant subspaces allows the dynamic compensator with the minimal unassignable dynamics to be straightforwardly derived. A steering along zeros technique is devised to guarantee decoupling with stability also in the presence of unstable unassignable dynamics of the minimal self-bounded controlled invariant.**

***Index Terms*—Exact decoupling, geometric approach, linear systems, preview.**

I. INTRODUCTION

In control problems, the system external inputs are references and disturbances usually assumed to be unknown. A feedback structure of the control system guarantees the best performance when those inputs

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G. Marro and E. Zattoni are with the Dipartimento di Elettronica, Informatica e Sistemistica, Università di Bologna, 40136 Bologna, Italy (e-mail: gmarro@deis.unibo.it; ezattoni@deis.unibo.it).

D. Prattichizzo is with the Dipartimento di Ingegneria dell'Informazione, Università di Siena, 53100 Siena, Italy (e-mail: prattichizzo@dii.unisi.it).

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are unknown. However, in many situations, the signals to be tracked or rejected are accessible for measurement or known with finite, or even infinite, preview. In those cases, better performance is achieved by exploiting information on the future of the signals by means of feedforward actions. Similarly, in the dual context of observation, some delay in reconstruction is often admissible.

As to the theoretic background of preview control and fixed-lag smoothing, many contributions separately dealing with the two problems exist [1]–[7]. In this note, we assume decoupling with preview as the basic problem. First, we show how to reduce tracking of previewed signals to decoupling by redefining the controlled variables. Then, we prove that unknown-input state observation and left-inversion, both with fixed lag, can be reduced to decoupling and tracking, both with preview. The geometric approach fosters a unified treatment of these classes of problems.

The second unifying contribution provided in this note concerns independence of the stabilizability condition from the signal to be decoupled being either unaccessible, or measurable, or known with finite preview. While the structural condition involves inclusion of the image of the external input matrix in larger and larger subspaces ($\mathcal{H} \subseteq \mathcal{V}^*$ [8], [9], $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$ [10], and $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}^*$ [11], [12]), the stabilizability condition (i.e., condition ii) of Theorem 1), first introduced for unaccessible and measurable signal decoupling [13]–[16], is shown to be valid in all cases. Moreover, if the structural condition holds but the stabilizability condition does not, it is shown that it is nonetheless possible to achieve decoupling of the external input with internal stability, provided that it is known in advance with infinite preview. Actually, infinite preview is not strictly necessary: a preview sufficiently greater than the greatest time constant associated to the unstable unassignable internal eigenvalues of $\mathcal{V}_m$, which are a subset of the plant invariant zeros, enables the problem to be solved with arbitrary accuracy.

Finally, to state the stabilizability condition, we exploit the properties of $\mathcal{V}_m$, the minimal internally stabilizable $(A, \mathcal{B})$-controlled invariant self-bounded with respect to $\mathcal{C}$ [13], [14], instead of those of $\mathcal{V}_g^*$, the maximal internally stabilizable $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$, often used in the literature [8], [9], [12]. Since an internally stabilizable $\mathcal{V}_m$ is contained in $\mathcal{V}_g^*$ [13], using $\mathcal{V}_m$ allows us to reduce the state dimension of the decoupling controller.

II. Main Results

The discrete time-invariant linear system

$$x(k+1) = A\,x(k) + B\,u(k) + H\,h(k) \tag{1}$$
$$y(k) = C\,x(k) \tag{2}$$

is considered, with state $x \in \mathcal{X} = \mathbb{R}^n$, control input $u \in \mathbb{R}^p$, controlled output $y \in \mathbb{R}^q$, external input $h \in \mathbb{R}^s$ (either unaccessible, or measurable, or known with preview). B, H, C are assumed to be full rank. $\mathcal{B}$, $\mathcal{H}$, $\mathcal{C}$ stand for $\operatorname{im} B$, $\operatorname{im} H$, $\ker C$, respectively. Some geometric concepts are recalled by using the notation of [16]. A subspace $\mathcal{V}$ is an internally stabilizable $(A, \mathcal{B})$-controlled invariant if a matrix F exists such that $(A+BF)\,\mathcal{V} \subseteq \mathcal{V}$ and $\sigma((A+BF)|_{\mathcal{V}}) \subset \mathbb{C}^{\odot}$, where $\mathbb{C}^{\odot}$ is the open unit disk in the complex plane. A subspace $\mathcal{S}$ is an externally stabilizable $(A, \mathcal{C})$-conditioned invariant if G exists such that $(A+GC)\,\mathcal{S} \subseteq \mathcal{S}$ and $\sigma((A+GC)|_{\mathcal{X}/\mathcal{S}}) \subset \mathbb{C}^{\odot}$, where $\mathcal{X}/\mathcal{S}$ is the quotient space of $\mathcal{X}$ over $\mathcal{S}$. The maximal $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$ is denoted by $\max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$ or, briefly, $\mathcal{V}^*$, the minimal $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{B}$ is denoted by $\min \mathcal{S}(A, \mathcal{C}, \mathcal{B})$ or $\mathcal{S}^*$. $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$ is the constrained reachability subspace on $\mathcal{V}^*$. The triple (A, B, C) is left-invertible if $\mathcal{V}^* \cap \mathcal{S}^* = \{0\}$, right-invertible if $\mathcal{V}^* + \mathcal{S}^* = \mathbb{R}^n$. The invariant zeros of (A, B, C) are the unassignable internal eigenvalues of $\mathcal{V}^*$, i.e., $\mathcal{Z} = \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}})$. If (A, B, C) is right-invertible, its relative degree is the least integer ρ such that $C\,\mathcal{S}_\rho = \mathbb{R}^q$, where the subspaces $\mathcal{S}_i$, $i = 1, 2, \ldots, \rho_M$, are given by the algorithm $\mathcal{S}_1 = \mathcal{B}$, $\mathcal{S}_i = A\,(\mathcal{S}_{i-1} \cap \mathcal{C}) + \mathcal{B}$, with ρ_M being the least integer such that $\mathcal{S}_{\rho_M+1} = \mathcal{S}_{\rho_M}$. If (A, B, C) is right-invertible and left-invertible, its relative degree is the number of steps for evaluating $\mathcal{S}^*$, i.e., $\rho = \rho_M$ and $\mathcal{S}_\rho = \mathcal{S}^*$. The following theorem, proved in [13]–[15], refers to unaccessible signals $h(k)$.

Theorem 1 (Unaccessible Signal Decoupling): If the input $h(k)$ to be decoupled in system (1),(2) is unaccessible, a bounded control law $u(k)$ decoupling the signal $h(k)$ exists if and only if: i) $\mathcal{H} \subseteq \mathcal{V}^*$; ii) $\mathcal{V}_m = \mathcal{V}^* \cap \min\,\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H})$ is internally stabilizable.

Although the cases of h measurable and previewed have been extensively studied from the structural point of view, a complete theory of the problem with stability is not yet available. In fact, while it is well-known that condition *i* of Theorem 1 modifies into $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$ in measurable signal decoupling [10] and into $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}^*$ in previewed signal decoupling [11], [12], it is shown herein for the first time that the same stabilizability condition holds not only for unaccessible and measurable signal decoupling [15], [16], but also for signals known with finite preview. The block diagram for measurable and previewed signal decoupling is shown in Fig. 1. Σ is the system (1), (2), Σ_c is the feedforward compensator, the block k_p-*delay* is a cascade of k_p unit delays on the input h signal flow modeling the preview of k_p steps (if h is measurable, k_p is set to zero).

Problem 1 (Previewed Signal Decoupling): Refer to Fig. 1. Let Σ be ruled by (1), (2). Let $\sigma(A) \subset \mathbb{C}^{\odot}$. Let $x(0) = 0$. Let $h(k)$ be known with a finite preview of k_p steps, with $k_p \geq \rho_M$. Design a bounded-input–bounded-output (BIBO) stable feedforward compensator Σ_c, having $h_p(k) = h(k+k_p)$ as input and $u(k)$ as output, such that $y(k)$ is identically zero.

Lemma 1: For any $\mathcal{Q} \subseteq \mathbb{R}^n$

$$\min\,\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{Q}) = \min\,\mathcal{S}(A, \mathcal{C}, \mathcal{S}^* + \mathcal{Q}).$$

Proof: By construction, the subspaces generated by the standard algorithms for the minimum $(A, \mathcal{C})$-conditioned invariants, respectively, containing $\mathcal{B} + \mathcal{Q}$ and $\mathcal{B}$ satisfy the inclusions

$$\mathcal{S}'_1 = \mathcal{B} + \mathcal{Q} \supseteq \mathcal{S}_1 = \mathcal{B}$$
$$\mathcal{S}'_i = A(\mathcal{S}'_{i-1} \cap \mathcal{C}) + \mathcal{B} + \mathcal{Q} \supseteq \mathcal{S}_i = A(\mathcal{S}_{i-1} \cap \mathcal{C}) + \mathcal{B}$$

with $i = 2, 3, \ldots, \rho_M$, where ρ_M is the number of steps for evaluating $\mathcal{S}^*$. These algorithms do not necessarily converge in the same number of steps, but the last inclusion implies $\min\,\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{Q}) \supseteq \mathcal{S}^*$. Hence, it implies $\min\,\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{Q}) \supseteq \mathcal{S}^* + \mathcal{B} + \mathcal{Q} \supseteq \mathcal{S}^* + \mathcal{Q}$. In the light of the latter inclusion, $\min\,\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{Q})$ is an $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{S}^* + \mathcal{Q}$, which implies $\min\,\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{Q}) \supseteq \min\,\mathcal{S}(A, \mathcal{C}, \mathcal{S}^* + \mathcal{Q})$. Finally, $\min\,\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{Q}) \subseteq \min\,\mathcal{S}(A, \mathcal{C}, \mathcal{S}^* + \mathcal{Q})$ follows from $\mathcal{B} + \mathcal{Q} \subseteq \mathcal{S}^* + \mathcal{Q}$ and completes the proof. ■

Theorem 2 (Previewed Signal Decoupling): Problem 1 is solvable if and only if: i) $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}^*$; ii) $\mathcal{V}_m = \mathcal{V}^* \cap \min\,\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H})$ is internally stabilizable.

Proof: Condition i) is well settled in the literature and its proof is omitted [11], [12]. To understand the role of $\mathcal{V}^*$ and $\mathcal{S}^*$ the reader is referred to Section III and in particular to Algorithm 1 for $\mathcal{S}^*$ and Algorithm 2 for $\mathcal{V}^*$. The rest of the proof concerns condition ii).

If: First note that, owing to condition i), subspaces $\mathcal{H}_{\mathcal{S}^*} \subseteq \mathcal{S}^*$ and $\mathcal{H}_{\mathcal{V}^*} \subseteq \mathcal{V}^*$ exist such that $\mathcal{H} = \mathcal{H}_{\mathcal{S}^*} + \mathcal{H}_{\mathcal{V}^*}$. Because of linearity, with no loss of generality, consider the signal h as a unit pulse (δ) signal $h(k) = e_i\,\delta(k - \rho_M)$, with $k = 0, 1, \ldots$ and e_i ($i = 0, 1, \ldots, s$) denoting the generic ith vector of the main basis of $\mathbb{R}^s$. The input $h(k)$

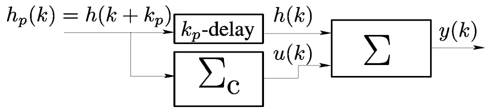

Fig. 1. Block diagram for previewed signal decoupling.

is assumed to be previewed by ρ_M time instants. Let ξ be defined as $\xi = H\, e_i\, \delta(k - \rho_M)$. Then, ξ can be expressed as $\xi = \xi_{\mathcal{S}^*} + \xi_{\mathcal{V}^*}$ with $\xi_{\mathcal{S}^*} \in \mathcal{H}_{\mathcal{S}^*}$ and $\xi_{\mathcal{V}^*} \in \mathcal{H}_{\mathcal{V}^*}$. The decomposition of ξ into $\xi_{\mathcal{S}^*}$ and $\xi_{\mathcal{V}^*}$ is not unique if $\mathcal{H}_{\mathcal{S}^*} \cap \mathcal{H}_{\mathcal{V}^*} \neq \{0\}$, which may occur if the system is nonleft-invertible, but the arguments herein presented hold for any decomposition considered. By definition of $\mathcal{S}^*$, any state belonging to $\mathcal{H}_{\mathcal{S}^*}$ can be reached from the origin in ρ_M steps, at most, along a trajectory belonging to $\mathcal{C}$, therefore invisible at the output until the last step but one. Hence, the component $\xi_{\mathcal{S}^*}$ can be nulled by applying the control input sequence driving the state from the origin to its opposite, $-\xi_{\mathcal{S}^*}$. On the other hand, the component $\xi_{\mathcal{V}^*}$ can be localized on $\mathcal{V}^*$, since both the conditions of Theorem 1 are satisfied. In fact, $\mathcal{H}_{\mathcal{V}^*} \subseteq \mathcal{V}^*$ by construction, and $\mathcal{V}^* \cap \min\, \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_{\mathcal{V}^*})$ is internally stabilizable since, by Lemma 1

$$\begin{aligned}\mathcal{V}^* &\cap \min\, \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_{\mathcal{V}^*})\\ &= \mathcal{V}^* \cap \min\, \mathcal{S}(A, \mathcal{C}, \mathcal{S}^* + \mathcal{H}_{\mathcal{V}^*})\\ &= \mathcal{V}^* \cap \min\, \mathcal{S}(A, \mathcal{C}, \mathcal{S}^* + \mathcal{H})\\ &= \mathcal{V}^* \cap \min\, \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H})\\ &= \mathcal{V}_m\end{aligned}$$

and $\mathcal{V}_m$ is internally stabilizable by assumption.

Only If: If $\mathcal{V}_m$ is not internally stabilizable then, from the equalities above, $\mathcal{V}^* \cap \min\, \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_{\mathcal{V}^*})$ is not internally stabilizable and, by virtue of Theorem 1, it is not possible to decouple components of signal $h(k)$ lying on $\mathcal{V}^*$ and not belonging to $\mathcal{S}^*$. ∎

Remark 1: The assumption that A is stable is not restrictive with respect to those of stabilizability of (A, B) and detectability of (A, C). On those assumptions, a stable system can be obtained by output dynamic feedback while preserving minimality of the compensator unassignable dynamics [17].

Remark 2: If the initial condition is not zero in Problem 1, there is a transient of the output $y(k)$ decaying to zero with the internal modes of Σ, which is stable by assumption. If Σ is not stable originally, but it is stabilizable and detectable, the internal modes are those of the stabilized system (Remark 1).

In the following, Theorem 2 is extended to the context of fixed-lag observation by duality. Hence, the observer which solves the following Problem 2 can be derived from the compensator solving Problem 1.

The discrete time-invariant linear system

$$x(k+1) = A\, x(k) + B\, u(k) \tag{3}$$
$$y(k) = C\, x(k) \tag{4}$$
$$e(k) = E\, x(k) \tag{5}$$

is considered, with state $x \in \mathbb{R}^n$, unaccessible input $u \in \mathbb{R}^p$, and informative output $y \in \mathbb{R}^q$. Let $e \in \mathbb{R}^s$ be the linear combination of the states to be observed. Let B, C, E be full rank. Let $\mathcal{E}$ stand for $\ker E$. Fig. 2 shows the block diagram for unknown-input observation of a linear combination of the states with fixed lag. Σ stands for the system (3),(4),(5), Σ_e for the observer, and the block k_p*-delay* for a cascade of k_p unit delays on the unaccessible output e signal flow to simulate the delay of k_p steps accepted in observation.

Problem 2 (Fixed-Lag Unknown-Input State Observation): Refer to Fig. 2. Let Σ be ruled by (3),(4),(5). Let $\sigma(A) \subset \mathbb{C}^{\odot}$. Let $x(0) = 0$. Let the unaccessible output $e(k)$ to be observed with a finite delay of k_p steps, with $k_p \geq \rho_M$, i.e., the signal to be observed is $e_d(k) = e(k - k_p)$. Design a BIBO stable dynamic observer Σ_e, with $y(k)$ as input and $-\hat{e}_d(k)$ as output, such that $\eta_d(k) = e_d(k) - \hat{e}_d(k)$ is identically zero.

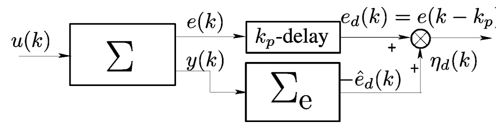

Fig. 2. Block diagram for fixed-lag unknown-input state observation.

Theorem 3 (Fixed-Lag Unknown-Input State Observation): Problem 2 is solvable if and only if: i) $\mathcal{E} \supseteq \mathcal{V}^* \cap \mathcal{S}^*$; ii) $\mathcal{S}_{\mathrm{MAX}} = \mathcal{S}^* + \max\, \mathcal{V}(A, \mathcal{B}, \mathcal{C} \cap \mathcal{E})$ is externally stabilizable.

Proof: It follows by applying basic duality arguments to the involved subspaces [16]. ∎

Stable right inversion with preview and stable left inversion with fixed lag are encompassed in our unified framework as is shown later. Problems 3 and 4 are stated for the discrete time-invariant linear system

$$x(k+1) = A\, x(k) + B\, u(k) \tag{6}$$
$$y(k) = C\, x(k) \tag{7}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$, output $y \in \mathbb{R}^q$. B and C are assumed to be full rank. The input u is the control input in Problem 3, the unknown input in Problem 4. The output y is the controlled output in Problem 3, the informative output in Problem 4. The triple (A, B, C) is assumed to be right-invertible in Problem 3, left-invertible in Problem 4.

Problem 3 (Stable Right Inversion With Preview): Refer to Fig. 3. Let Σ be ruled by (6) and (7). Let $\sigma(A) \subset \mathbb{C}^{\odot}$. Let $x(0) = 0$. Let the reference signal $h(k)$, with $h \in \mathbb{R}^q$, be previewed by k_p steps, with $k_p \geq \rho_M$. Design a BIBO stable feedforward compensator Σ_c, with $h_p(k) = h(k + k_p)$ as input and $u(k)$ as output, such that the output $y(k)$ is identically equal to the reference signal $h(k)$.

According to the block diagram shown in Fig. 3, Problem 3 is solvable as a previewed signal decoupling problem stated for a modified plant $\tilde{\Sigma}$ ruled by $x(k+1) = A\, x(k) + B\, u(k)$, $\tilde{y}(k) = C\, x(k) - h(k)$. The presence of the feedthrough term can be managed by resorting to a well-known contrivance presented, e.g., in [16].

Problem 4 (Stable Left Inversion With Fixed Lag): Refer to Fig. 4. Let Σ be ruled by (6),(7). Let $\sigma(A) \subset \mathbb{C}^{\odot}$. Let $x(0) = 0$. The unknown input $u(k)$ must be observed with a delay of k_p steps, with $k_p \geq \rho_M$, i.e., the signal to be observed is $u_d(k) = u(k - k_p)$. Design a BIBO stable dynamic unknown-input observer Σ_e, with $y(k)$ as input and $\hat{e}_d(k)$ as output, such that $\hat{e}_d(k)$ is identically equal to the delayed unknown input $u_d(k)$.

As shown in Fig. 4, Problem 4 can be reduced to state observation in the presence of unknown inputs for a modified plant $\tilde{\Sigma}$ ruled by $x(k+1) = A\, x(k) + B\, u(k)$, $y(k) = C\, x(k)$, $e(k) = u(k)$.

III. Previewed Signal Decoupling: Compensator Design

In the previous section, a necessary and sufficient condition for decoupling with stability of signals which are known with finite preview has been provided (Theorem 2). However, the stabilizability condition (namely, condition *ii*) can be relaxed to the condition that $\mathcal{V}_m$ does not have any unassignable internal eigenvalue on the unit circle if the signal to be decoupled is completely known in advance. The case of infinite preview, formerly investigated by the authors in the context of right inversion for nonminimum-phase systems [5], is extended to decoupling herein.

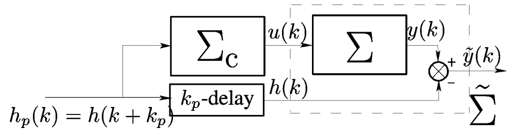

Fig. 3. Block diagram for stable right inversion with preview.

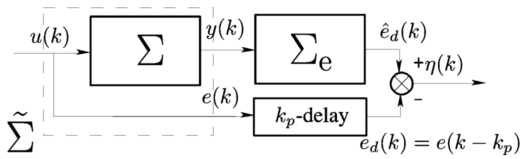

Fig. 4. Block diagram for stable left inversion with fixed lag.

In this section, we will introduce an algorithmic solution to Problem 1 where the structural condition of Theorem 2 is assumed to be satisfied and two different control strategies are devised according to whether the stabilizability condition is satisfied or not. The synthesis of the precompensator, which includes a finite impulse response system in the former case and a convolutor with an infinitely large window in the latter one (in practice, however, it is approximated by a finite impulse response system with a sufficiently large window to make the truncation error negligible), is based on the following considerations.

The subspace $\mathcal{V}_m$ is a locus of initial states in $\mathcal{C}$ corresponding to trajectories controllable on $\mathcal{C}$, while $\mathcal{S}^*$ is the maximum set of states that can be reached from the origin in ρ_M steps along trajectories with all the intermediate states in $\mathcal{C}$. Then, suppose that an impulse is applied to the input h at the time ρ_M, thus producing a component of the state $x_h \in \mathcal{H}$, which is decomposable as $x_h = x_{h,S} + x_{h,V}$, with $x_{h,S} \in \mathcal{S}^*$ and $x_{h,V} \in \mathcal{V}_m$ (note that $\mathcal{H} \subseteq \mathcal{V}_m + \mathcal{S}^*$ is implied by the structural condition [13], [14]). The component $x_{h,S}$ can be nulled by applying the control sequence that drives the state from the origin to $-x_{h,S}$ along a trajectory in $\mathcal{S}^*$. The component $x_{h,V}$ can be maintained on $\mathcal{V}_m$ by a suitable control action in the time interval $\rho_M \le k < \infty$ while avoiding state divergence, if all the internal unassignable modes of $\mathcal{V}_m$ are stable (or stabilizable). Otherwise, $x_{h,V}$ must be further decomposed as $x_{h,V} = x_{h,V_S} + x_{h,V_U}$, with x_{h,V_S} belonging to the subspace of the stable (or stabilizable) internal modes of $\mathcal{V}_m$ and x_{h,V_U} belonging to that of the unstable modes. The former component can be maintained on $\mathcal{V}_m$, avoiding state divergence, by a suitable control action over the time interval $\rho_M \le k < \infty$, while the latter can be nulled by reaching $-x_{h,V_U}$ with a control action, applied over the time interval $-\infty < k \le \rho_M - 1$, corresponding to a trajectory in $\mathcal{V}_m$ from the origin.

The following Algorithms 1 and 2 provide the control and state sequences for motions on $\mathcal{S}^*$ and $\mathcal{V}_m$, respectively, assuming $h(k) = I\,\delta(k - \rho_M)$. This particular choice of the input h directly yields the FIR system convolution profiles and the matrices of the dynamic unit. Algorithms 1 and 2 require that the system (1), (2) be left-invertible: Algorithm 3 provides a means to deal with nonleft-invertible systems.

Before introducing the abovementioned algorithms, a particular state space basis transformation is performed. The matrix H must be decomposed as $H = VH_1' + SH_2'$, where V and S denote basis matrices of $\mathcal{V}_m$ and $\mathcal{S}^*$, respectively. Let F be such that $(A + BF)\,\mathcal{V}_m \subseteq \mathcal{V}_m$ and let $T = [V\ S\ T_1]$ be the state space basis transformation considered. The system matrices in the new basis have the structures

$$A_F' = \begin{bmatrix} A_{11}' & A_{12}' & A_{13}' \\ O & A_{22}' & A_{23}' \\ O & A_{32}' & A_{33}' \end{bmatrix} \tag{8}$$

$$B' = [O \quad B_2'^{\top} \quad O]^{\top},\ H' = [H_1'^{\top} \quad H_2'^{\top} \quad O]^{\top} \tag{9}$$

$$C' = [O \quad C_2' \quad C_3'],\ F' = [F_1' \quad F_2' \quad F_3']. \tag{10}$$

Algorithm 1 (Motion on $\mathcal{S}^*$): The controls $U_1(k)$, with $k = 0, \ldots, \rho_M - 1$, and the states $X_1(k)$, with $k = 1, \ldots, \rho_M$, are derived through the following steps.

1. Compute basis matrices M_i of the subspaces $\mathcal{S}_i \cap \mathcal{C}$ for $i = 1, \ldots, \rho_M - 1$.
2. Compute the sequences $\beta(i)$ and $U_1(i)$, $i = 1, \ldots, \rho_M - 1$, as

$$\begin{bmatrix} \beta(\rho_M - j) \\ U_1(\rho_M - j) \end{bmatrix} = [A\,M_{\rho_M - j} \quad B]^{\#} \quad M_{\rho_M - j+1}\beta(\rho_M - j + 1)$$

with $j = 1, \ldots, \rho_M - 1$, $M_{\rho_M} = S$, and $\beta(\rho_M) = -H_2'$.
3. Compute $U_1(0)$ driving the states from the origin to $M_1\beta(1)$ as $U_1(0) = B^{\#}M_1\,\beta(1)$.
4. Compute the states $X_1(i)$, $i = 1, \ldots, \rho_M$, as $X_1(i) = M_i\,\beta(i)$, $i = 1, \ldots, \rho_M$. □

Algorithm 2 (Motion on $\mathcal{V}_m$): Two different strategies are set forth, depending on whether $\mathcal{V}_m$ is internally stabilizable or not.

1) If $\mathcal{V}_m$ is internally stabilizable, the motion on $\mathcal{V}_m$ is provided by the pair (A_{11}', H_1') in (8),(9): i.e., the states restricted to $\mathbb{R}^{n_\mathcal{V}}$, $n_\mathcal{V} = \dim(\mathcal{V}_m)$, are $X_2(\rho_M + i) = (A_{11}')^i\,H_1'$, $i = 0, 1, \ldots$, and the controls are $U_2(\rho_M + i) = F_1'\,(A_{11}')^i\,H_1'$, $i = 0, 1, \ldots$.

2) If $\mathcal{V}_m$ is not internally stabilizable, a second state space basis transformation T', whose aim is to separate the stable and unstable modes of $\mathcal{V}_m$, is required. The matrices A_{11}'', H_1'', and F_1'', respectively corresponding to A_{11}', H_1' and F_1' in the new basis, have the structures

$$A_{11}'' = \begin{bmatrix} A_S & O \\ O & A_U \end{bmatrix}\ H_1'' = \begin{bmatrix} H_S \\ H_U \end{bmatrix}\ F_1'' = \begin{bmatrix} F_S^{\top} \\ F_U^{\top} \end{bmatrix}^{\top}.$$

An infinite preaction, nulling the unstable component of the state H_U at the time instant ρ_M, must be computed backward through the matrix A_U. The infinite states restricted to $\mathbb{R}^{n_u}$, $n_u = \dim(\mathcal{V}_m^U)$, are $X_3(\rho_M - j) = -A_U^{-j}H_U$, $j = 0, 1, \ldots$, and the infinite controls are $U_3(\rho_M - j) = -F_U A_U^{-j} H_U$, $j = 1, 2, \ldots$. A convolutor with an infinitely large window is needed to reproduce the infinite controls $U_3(\rho_M - j)$ computed backward. As mentioned previously, the convolutor window is choosen large enough to make the truncation error negligible in practice. The stable component of the state H_S is managed as in case of $\mathcal{V}_m$ stabilizable. □

In the light of the previous algorithms, we conclude that, if all the internal modes of $\mathcal{V}_m$ are stable, decoupling is achieved by finite preaction (dead-beat, motion on $\mathcal{S}^*$) and postaction (motion on $\mathcal{V}_m$ along the stable zeros). The first is realized by feeding the controlled system with the output of a ρ_M-step FIR system, the latter is realized as the output of a stable dynamic unit. Hence, the compensator is the parallel connection of a ρ_M-step FIR system and a dynamic unit. The IO equation of the FIR system is

$$u_F(k) = \sum_{\ell=0}^{\rho_M - 1} \Phi(\ell)\,h(k - \ell), \qquad k = 0, 1, \ldots \tag{11}$$

with $\Phi(\ell) = U_1(\ell)$, $\ell = 0, \ldots, \rho_M - 1$. The equations of the dynamic unit are

$$w(k+1) = N\, w(k) + L\, h(k - \rho_M), \qquad k = 0, 1, \ldots \quad (12)$$

$$u_D(k) = M\, w(k) \quad (13)$$

where $N = A'_{11}$, $L = H'_1$, $M = F'_1$. Hence, the control input is $u(k) = u_F(k) + u_D(k)$, $k = 0, 1, \ldots$. Otherwise, if unstable modes are present in $\mathcal{V}_m$, also infinite preaction is required. Since the evolution of the state according to the unstable modes of $\mathcal{V}_m$ can only be computed backward and reproduced through an FIR system, the FIR system window is enlarged to include the preaction time k_a, where this latter should be large enough to guarantee that the truncation error does not have any relevant effect. In this case, the compensator is the parallel connection of a $(k_a + \rho_M)$-step FIR system and a dynamic unit. Equation (11) is modified into

$$u_F(k) = \sum_{\ell=-k_a}^{\rho_M - 1} \Phi(\ell)\, h(k - \ell), \qquad k = 0, 1, \ldots$$

with $\Phi(\ell) = U_1(\ell) + U_3(\ell)$, $\ell = -k_a, \ldots, \rho_M - 1$ (with a slight abuse of notation the control sequences are assumed to be zero wherever they are not explicitly defined). The dynamic unit is described by (12) and (13) with $N = A_S$, $L = H_S$, $M = F_S$.

If the triple (A, B, C) is nonleft-invertible, the previous procedure can be applied anyhow, provided that a preliminary manipulation is performed to obtain a left-invertible triple and the results thus obtained are adapted to the original system.

Algorithm 3 (Extension to Non Left-Invertible Systems): If the triple (A, B, C) is nonleft-invertible, the previous procedure must be applied to (A^*, B^*, C), with

1) $A^* = A + BF^*$, where F^* is such that $(A + BF^*)\, \mathcal{V}^* \subseteq \mathcal{V}^*$ and $\sigma((A + BF^*)|_{\mathcal{R}_{\mathcal{V}^*}}) \subset \mathbb{C}^{\odot}$;
2) $B^* = B\, U^*$, where U^* is a basis matrix of the subspace $\mathcal{U}^* = (B^{-1}\, \mathcal{V}^*)^{\perp}$, the orthogonal complement of the inverse image of $\mathcal{V}^*$ with respect to B.

Let $\bar{U}_i(k)$ and $\bar{X}_i(k)$, with $i = 1, 2, 3$ and k consistently defined, be the sequences of controls and states provided by Algorithms 1 and 2 applied to (A^*, B^*, C). The corresponding control sequences for (A, B, C) must be computed as $U_i(k) = U^* \bar{U}_i(k) + F^* \bar{X}_i(k)$, $i = 1, 2, 3$. □

IV. Conclusion

Theoretical results have been presented to solve decoupling and tracking with preview and the dual counterparts, i.e., unknown-input state observation and unknown-input observation with fixed lag, in a unified framework. Decoupling with preview is assumed as the basic problem and it is solved in the geometric context by exploiting the properties of the minimal self-bounded controlled invariant subspace satisfying the structural constraint. A complete algorithmic procedure has been detailed to design the compensator both in the case where the stabilizability condition is satisfied, and in the case where unstable unassignable internal eigenvalues of the minimal self-bounded controlled invariant are present.

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