On a Class of Hierarchical Formations of Unicycles and Their Internal Dynamics

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Abstract—This paper studies a class of hierarchical formations for an ordered set of $n + 1$ unicycle robots: the first robot plays the role of the leader and the formation is induced through a constraint function F , so that the position and orientation of the i th robot depends only on the pose of the preceding ones. We study the dynamics of the formation with respect to the leader's reference frame by introducing the concept of *reduced internal dynamics*, we characterize its equilibria and provide sufficient conditions for their existence. The discovered theoretical results are applied to the case in which the constraint F induces a formation where the i th robot follows a convex combination of the positions of the previous $i - 1$ vehicles. In this case, we prove that if the curvature of the leader's trajectory is sufficiently small, the positions and orientations of the robots, relative to the leader's reference frame, are confined in a precise polyhedral region.

Index Terms—Formation control, mobile robots, motion control, multiagent systems, nonlinear systems.

I. INTRODUCTION

OVER the past few years, multi-agent systems and cooperative control have been the subject of extensive research in the control and robotics communities [1]–[5]. The reasons for this success must be sought in multiple factors, such as, e.g., in the wider availability of wireless communication technologies and low-cost processing units, in the growing interest in parallel and embedded computing, in the definition of increasingly demanding tasks requiring the coordinate action of multiple autonomous agents to be successfully accomplished.

Due to its wide applicability in real-world scenarios (e.g., in patrolling/reconnaissance, entrapping/escorting, map building and exploration missions), the *formation control problem* has attracted considerable attention in the multi-agent systems literature (a list of representative papers, yet far from being complete, is [6]–[12]). The idea behind formation control is that of

Fig. 1. If the initial configuration of a team of unicycles is the one shown in the figure, the vehicles cannot move as a single rigid body.

controlling the relative position and orientation of the agents in a group, while allowing the group to move as a whole [13].

Three major research lines in formation control have been developed in the last decade: *behavior-based* [14], [15], *virtual structures* [16]–[18], and *leader-follower* [13], [19]–[21].

Recently, the interest in formation control has been awakened by the introduction of the original notion of rigidity. Moving from the seminal papers [22], [23], Anderson and coworkers have started to systematically apply the *rigid graph theory* [24] to the analysis of formations of autonomous agents and they have shown the relevance of the rigidity concept in sensor network localization problems [25] as well as in other branches of engineering [26]–[28].

In [29] and [30], graph rigidity ideas have been used to design decentralized gradient control laws for the stabilization of a group of kinematic points to a target formation. Additionally, in [31], a distributed algorithm that stabilizes the shape of a relative sensing network to a desired formation has been proposed: the algorithm relies on the global minimization of the “stress majorization function” (a tool from multidimensional scaling theory) associated to the network.

Note that differently from formations of fully actuated agents as those considered in [29]–[31], given a team of unicycle robots, it is in general impossible to guarantee that the vehicles move as a single rigid body for an arbitrary initial condition, since the nonholonomic constraints must be satisfied. This problem is illustrated in Fig. 1, where the robots have different orientations and they cannot move as a rigid body since this would violate the nonholonomic constraints.

In this respect, the study of the relative dynamics of the robots, is of paramount importance to prevent undesired phenomena, such as uncontrolled oscillations or instabilities in the formation shape when the robots must obey some coordination rule during their motion.

Dealing with the problem of unit-speed unicycles coordination and assuming an all-to-all communication, the authors in

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[32] have shown an interesting connection between coupled oscillator theory (and notably Kuramoto model of coupled oscillators [33]) and control of multiple nonholonomic vehicles. The results in [32] have been extended in several directions in recent years. In [34], the case of limited communication topology is considered and in [35] the design methodology developed in [32] is adapted to the trajectory-tracking case. Finally, in [36] the authors have investigated the effect of few leader agents on the controllability of a heterogeneous group of unit-speed unicycles obeying a controlled sinusoidal phase-coupling protocol.

In this paper, we study a class of hierarchical formations for an ordered set of $n + 1$ unicycle robots, where the first robot plays the role of the leader. The desired formation is obtained by imposing that the positions and orientations of the robots belong to a constraint which is the zero set of an assigned map F . Actually, the idea of defining a robot formation through the enforcement of a constraint is not new in the literature. In fact, there are similarities between the constraint function F defined in this paper and the *formation constraint function* considered in [17], the concept of formation constraints in [7], and the *abstraction* and the *goal function* introduced in [37] and [12], respectively.

With respect to the above references, we make some extra assumptions on the constraint F that allow the induced formation to possess some specific properties. In particular, to consider hierarchical formations, we assume that F belongs to a family of maps in which the variables are the positions and the orientations of the robots, and the state of the i th robot depends only on the states of the preceding ones. This may represent a unifying approach to study formations of unicycles with hierarchical structures and allow to find common properties in different control scenarios.

We present conditions on the map F which guarantee that, for any trajectory followed by the leader, there exist unique controls for the followers (i.e., unique linear and angular velocities) such that they are asymptotically in formation (i.e., the constraint function is asymptotically satisfied). In this way, the behavior of the formation is enforced by prescribing a constraint function F and the control law is entirely derived from it, and not assigned *a priori*.

Moreover, the dynamics of the formation with respect to the leader's reference frame is studied by introducing the concept of *reduced motion* of the follower robots, which describes the motion of the followers in the leader's frame. The evolution of the reduced motion is governed by the *reduced internal dynamics* system (note that this concept is similar to the use of "shape variables" in [38]).

An equilibrium point of the reduced internal dynamics represents a configuration in which the reduced motion is constant, that is the robots' relative positions and orientations are constant. This is actually the only case in which the formation moves as a single rigid body. The main contribution of this paper consists in characterizing these equilibrium configurations and in providing sufficient conditions for their existence, that depend only on the constraint function F and not on the specific choice of the control law.

In particular, a first theorem in Section II, states that if the vehicles are in formation, the relative position and the orientation of each follower with respect to the leader's reference frame

are constant if and only if the leader and the followers either move along circular paths or parallel straight lines. Moreover, a second theorem shows that an equilibrium configuration always exists if the leader moves along a circle of sufficiently small curvature or along a straight line.

The results of these two theorems are applied in Section III to a specific constraint function F that induces a formation where the i th robot follows a convex combination of the positions of the preceding $i - 1$ vehicles. This generalizes the hierarchical formations investigated in our previous works [39], [40]. For this specific case, a third theorem presents an invariance result for the internal dynamics of the formation, which provides sufficient conditions guaranteeing that, if the curvature of the leader's trajectory is sufficiently small, the positions and orientations of the follower robots, relative to the leader's reference frame, although not fixed, are confined in a precise polyhedral region.

Simulation results relative to the constraint function studied in Section III are presented in Section IV to illustrate the theory. Finally, in Section V, the main contributions of the paper are summarized and some concluding remarks are provided.

A. Notation

The following notation is used through the paper: S^1 denotes the quotient space $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, where $\mathbb{Z}$ is the set of the integer numbers; $\forall x = (x_1, x_2, \dots, x_n)^T$, $y = (y_1, y_2, \dots, y_n)^T \in \mathbb{R}^n$ ($n \geq 1$), $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$, $\|x\| = \sqrt{\langle x, x \rangle}$; $\forall \theta \in S^1$, $\tau(\theta) = (\cos \theta, \sin \theta)^T$, $\eta(\theta) = (-\sin \theta, \cos \theta)^T$; if $\bar{x} \in \mathbb{R}^n$, $B(\bar{x}, \epsilon) = \{x \in \mathbb{R}^n \mid \|x - \bar{x}\| \leq \epsilon\}$; for any $(x, y)^T \in \mathbb{R}^2 \setminus \{(0, 0)^T\}$, $\arg((x, y)^T)$ is the unique $\theta \in]-\pi, \pi]$ such that $(x, y)^T = \|(x, y)^T\| \tau(\theta)$.

Given a differentiable manifold $\mathcal{M}$, $T_x \mathcal{M}$ denotes the tangent space of $\mathcal{M}$ at $x \in \mathcal{M}$; given two functions $f(t)$ and $g(t)$, $(f \circ g)(t) = f(g(t))$ indicates their composition. If Ω is an open subset of $\mathbb{R}^n$, $\bar{x} \in \Omega$ and $F : \Omega \rightarrow \mathbb{R}^m$ is a differentiable map in $\bar{x}$, we denote by $F'(\bar{x})$ the $m \times n$ Jacobian matrix of F at $\bar{x}$. Moreover, we say that $F \in \mathcal{C}^{1,1}(\Omega, \mathbb{R}^m)$ if $F \in \mathcal{C}^1(\Omega, \mathbb{R}^m)$ and F' is a Lipschitz map.

II. HIERARCHICAL FORMATIONS OF UNICYCLE ROBOTS

Consider the following definition of robot as a velocity-controlled unicycle model [39].

Definition 1: A function $r = (x, y, \theta)^T \in \mathcal{C}^1([0, +\infty[, \mathbb{R}^2 \times S^1)$ is called a **unicycle robot** (or simply a **robot**) if there exists a control function $(v, \omega)^T \in \mathcal{C}^0([0, +\infty[, \mathbb{R}^2)$ such that

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ \omega \end{pmatrix}. \quad (1)$$

For any $t \in [0, +\infty[$, $(x(t), y(t))^T$ represents the position of the robot at time t , $\theta(t)$ its heading, $\tau(\theta(t))$ the normalized velocity vector and $\eta(\theta(t))$ the normalized vector orthogonal to $\tau(\theta(t))$. Hence, $\{\tau(\theta(t)), \eta(\theta(t))\}$ represents the robot reference frame at time t (see Fig. 2).

The next definition introduces an ordered set of $n + 1$ robots, with $n \geq 1$.

Fig. 2. Variables of the unicycle robot.

Definition 2: Let

$$\begin{aligned} \mathcal{X} &= (\mathbb{R}^2 \times S^1)^{n+1} \\ &= \{ \xi = (\xi_0, \xi_1, \dots, \xi_n) \mid \xi_i = (x_i, y_i, \theta_i)^T \in \mathbb{R}^2 \times S^1, \\ &\quad \forall i = 0, \dots, n \}. \end{aligned}$$

A $\mathcal{C}^1$ map $\mathcal{R} : [0, +\infty[\rightarrow \mathcal{X}$, defined by

$$\mathcal{R}(t) = (r_0(t), r_1(t), \dots, r_n(t))$$

where $r_i = (x_i, y_i, \theta_i)^T$ is a robot $\forall i = 0, 1, \dots, n$, is called an $(n+1)$ -**tuple of robots**. The set $\mathcal{X}$ is called the **configuration space**, r_0 the **leader**, and $r_1, r_2, \dots, r_n$ the **followers**.

The following definition characterizes the class of constraints considered in this paper to define a formation of an $(n+1)$ -tuple of robots.

Definition 3: Let F be a map given by $F = (F_1, F_2, \dots, F_n)^T$, where $F_i \in \mathcal{C}^{1,1}(\mathcal{X}, \mathbb{R}^2)$, $\forall i = 1, \dots, n$. The map F is called a **hierarchical constraint function** if

$$\mathcal{F} = \{ \xi \in \mathcal{X} \mid F(\xi) = 0 \} \text{ is not empty and} \quad (2)$$

$$\partial_{\xi_j} F_i(\xi) = 0, \quad \forall \xi \in \mathcal{X}, i = 1, \dots, n-1, j = i+1, \dots, n \quad (3)$$

that is, every F_i depends only on $(\xi_0, \dots, \xi_i)$. For the sake of brevity, we henceforth refer to F simply as **constraint function**. The set $\mathcal{F}$ is called the **constraint set**: it is the set of the configurations ξ compatible with the constraint function F .

The following definition introduces the notion of formation of an $(n+1)$ -tuple of robots used throughout the paper.

Definition 4: Let F be a constraint function. An $(n+1)$ -tuple of robots $\mathcal{R}$ is said in F -**formation** if

$$F(\mathcal{R}(t)) = 0, \quad \forall t \geq 0 \quad (4)$$

that is, if $\mathcal{R}(t) \in \mathcal{F}$, $\forall t \geq 0$.

As illustrated in the example of Fig. 3, if an $(n+1)$ -tuple of robots is in F -formation, the motion of each follower depends only on the behavior of the preceding robots. This is a consequence of the triangular structure of the function F , where every component F_i depends only on the position and the orientation of the robots that have index less or equal than i . In this way, F -formations represent a generalization of standard leader-follower architectures, where the motion of each robot depends only on the behavior of one of the preceding vehicles (see for instance [40]).

To provide concise statements in the following, let us introduce the vectors $v_F = (v_1, v_2, \dots, v_n)^T$, $\omega_F = (\omega_1, \omega_2, \dots, \omega_n)^T$, where v_i, ω_i are the control inputs of

 Fig. 3. In an F -formation, the robots are ordered: the first robot r_0 is the leader and each of the followers r_1, r_2, r_3 bases its motion on the behavior of the preceding robots, as shown by the colored arrows.

the i th robot. In this way the kinematic equations of the $n+1$ robots can be written in the form

$$\dot{\xi} = g(\xi, v_0, \omega_0, v_F, \omega_F) \quad (5)$$

where $g = (g_0, g_1, \dots, g_n)^T$ is such that $\forall i = 0, \dots, n$:

$$g_i(\xi, v_0, \omega_0, v_F, \omega_F) = \begin{pmatrix} \cos \theta_i & 0 \\ \sin \theta_i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_i \\ \omega_i \end{pmatrix}.$$

The following proposition states that $\mathcal{F}$ is a manifold and is controlled invariant for the closed-loop system (5) under the key assumption (6) (see below). Condition (6) requires that the directional derivative of F_i along the two vector fields $(\cos \theta_i, \sin \theta_i, 0)^T$ and $(0, 0, 1)^T$ is full rank, $\forall i = 1, \dots, n$. As it will be shown in the proof, this guarantees that it is always possible to find feedback controls v_F, ω_F such that $\mathcal{F}$ is an invariant manifold for the closed-loop system (5).

Proposition 1: Let F be a constraint function and suppose that $\forall \xi \in \mathcal{F}$, $\forall i = 1, \dots, n$

$$\det(\cos \theta_i \partial_{x_i} F_i(\xi) + \sin \theta_i \partial_{y_i} F_i(\xi), \partial_{\theta_i} F_i(\xi)) \neq 0. \quad (6)$$

Then the following facts hold:

- 1) $\mathcal{F}$ is a differentiable manifold of dimension $n+3$.
- 2) For each $\xi \in \mathcal{F}$ and for each $(v_0, \omega_0)^T \in \mathbb{R}^2$, there exist unique controls $v_F(\xi, v_0, \omega_0), \omega_F(\xi, v_0, \omega_0)$ such that $g(\xi, v_0, \omega_0, v_F(\xi, v_0, \omega_0), \omega_F(\xi, v_0, \omega_0)) \in T_{\xi} \mathcal{F}$.
- 3) For any $(v_0, \omega_0)^T \in \mathcal{C}^0([0, +\infty[, \mathbb{R}^2)$, $\forall \bar{\xi} \in \mathcal{F}$, the following property holds: $\mathcal{R}$ is an $(n+1)$ -tuple of robots in F -formation such that $\mathcal{R}(0) = \bar{\xi}$ if and only if $\mathcal{R}$ is the unique solution of the following system:

$$\begin{cases} \dot{\xi} = g(\xi, v_0, \omega_0, v_F(\xi, v_0, \omega_0), \omega_F(\xi, v_0, \omega_0)) \\ \xi(0) = \bar{\xi}. \end{cases} \quad (7)$$

In particular, if the $(n+1)$ -tuple of robots is in F -formation at the initial time, then for any trajectory $(x_0, y_0, \theta_0)^T$ of the leader r_0 , there exists a unique choice of controls $(v_i, \omega_i)^T \in \mathcal{C}([0, +\infty[, \mathbb{R}^2)$, $i = 1, \dots, n$, such that $r_1, \dots, r_n$ follow r_0 in F -formation.

Proof: Let F' be the Jacobian matrix of F . Since $\text{rank } F'(\xi) = 2n$ by (6), we have that 1) holds by the implicit function theorem. To prove 2), set, $\forall i = 1, \dots, n$,

$\forall j = 0, \dots, n$, $\partial_{\xi_j} F_i(\xi) = (\partial_{x_j} F_i(\xi), \partial_{y_j} F_i(\xi), \partial_{\theta_j} F_i(\xi))$, which is a 2×3 matrix and $\forall i = 1, \dots, n$

$$A_{ij}(\xi) = \partial_{\xi_j} F_i(\xi) \begin{pmatrix} \cos \theta_j & 0 \\ \sin \theta_j & 0 \\ 0 & 1 \end{pmatrix} = (\cos \theta_j \partial_{x_j} F_i(\xi) + \sin \theta_j \partial_{y_j} F_i(\xi), \partial_{\theta_j} F_i(\xi)). \quad (8)$$

Then, $\forall (v_0, \omega_0)^T \in \mathbb{R}^2$, let $v_F(\xi, v_0, \omega_0)$, $\omega_F(\xi, v_0, \omega_0)$ be given by the unique solution of the following triangular system:

$$A_{ii}(\xi) \begin{pmatrix} v_i \\ \omega_i \end{pmatrix} = - \sum_{j=0}^{i-1} A_{ij}(\xi) \begin{pmatrix} v_j \\ \omega_j \end{pmatrix}, \quad i = 1, \dots, n \quad (9)$$

which is solvable since $\det A_{ii}(\xi) \neq 0$ by (8) and hypothesis (6). Therefore 2) holds. Finally, 3) is a consequence of 2) and the existence and uniqueness of the solution of (7). ■

Remark 1: From (9) it follows that the control functions $v_F(\xi, v_0, \omega_0) = (v_1, v_2, \dots, v_n)^T$, $\omega_F(\xi, v_0, \omega_0) = (\omega_1, \omega_2, \dots, \omega_n)^T$, appearing in 2) of proposition 1, can be computed iteratively as follows:

$$\begin{aligned} \begin{pmatrix} v_1 \\ \omega_1 \end{pmatrix} &= -A_{11}^{-1}(\xi) A_{10}(\xi) \begin{pmatrix} v_0 \\ \omega_0 \end{pmatrix} \\ \begin{pmatrix} v_2 \\ \omega_2 \end{pmatrix} &= -A_{22}^{-1}(\xi) \left(A_{21}(\xi) \begin{pmatrix} v_1 \\ \omega_1 \end{pmatrix} + A_{20}(\xi) \begin{pmatrix} v_0 \\ \omega_0 \end{pmatrix} \right) \\ &\vdots \\ \begin{pmatrix} v_n \\ \omega_n \end{pmatrix} &= -A_{nn}^{-1}(\xi) \left(\sum_{j=0}^{n-1} A_{ij}(\xi) \begin{pmatrix} v_j \\ \omega_j \end{pmatrix} \right). \end{aligned} \quad (10)$$

Note that the controls defined by (10) are not decentralized since v_i, ω_i depend on the states of all preceding robots $\xi_j, j = 1, \dots, i$.

The following proposition deals with the stabilizability of the controlled invariant manifold $\mathcal{F}$. In particular, it shows that the regularity condition (11), which is slightly stronger than (6) (see also proposition 6 below), guarantees the local stabilizability of a robot formation. Roughly speaking, this means that if the robots are not initially “too far” from being in formation, we can always determine a control law for the followers that allows them to asymptotically achieve the desired formation, regardless of the trajectory of the leader.

Proposition 2: Suppose that F is a constraint function such that there exists $\epsilon > 0$, so that $\forall \xi \in \mathcal{F}(\epsilon), \forall i = 1, \dots, n$:

$$\det(\cos \theta_i \partial_{x_i} F_i(\xi) + \sin \theta_i \partial_{y_i} F_i(\xi), \partial_{\theta_i} F_i(\xi)) \neq 0 \quad (11)$$

where $\mathcal{F}(\epsilon) = \{\xi \in \mathcal{X} \mid \|F(\xi)\| \leq \epsilon\}$.

Then $\forall v_0, \omega_0 \in \mathcal{C}([0, +\infty[, \mathbb{R})$, there exist control functions $v_F(\xi, v_0, \omega_0), \omega_F(\xi, v_0, \omega_0) \in \mathcal{C}([0, +\infty[\times \mathcal{X}, \mathbb{R}^n)$ such that, $\forall \xi \in \mathcal{F}(\epsilon)$, the solution $\mathcal{R}(t)$ of the initial value problem (7) is defined on $[0, +\infty[$ and is such that $\lim_{t \rightarrow +\infty} \|F(\mathcal{R}(t))\| = 0$.

Proof: By hypothesis (11), using the notations of the proof of proposition 1, it suffices to define v_F, ω_F as the solution of the following system: $\forall i = 1, \dots, n$

$$A_{ii}(\xi) \begin{pmatrix} v_i \\ \omega_i \end{pmatrix} = - \sum_{j=0}^{i-1} A_{ij}(\xi) \begin{pmatrix} v_j \\ \omega_j \end{pmatrix} - k F_i(\xi) \quad (12)$$

with $k > 0$. Since $\det A_{ii}(\xi) \neq 0$, by (8) and hypothesis (6) the control functions are given by $\forall i = 1, \dots, n$

$$\begin{pmatrix} v_i \\ \omega_i \end{pmatrix} = A_{ii}^{-1}(\xi) \left(- \sum_{j=0}^{i-1} A_{ij}(\xi) \begin{pmatrix} v_j \\ \omega_j \end{pmatrix} - k F_i(\xi) \right). \quad (13)$$

Since F is $\mathcal{C}^{1,1}$ by definition 3, the right-hand side of the closed-loop system (7) is continuous in the couple (t, ξ) and globally Lipschitz as a function of ξ . Therefore, $\forall \xi \in \mathcal{F}(\epsilon)$, $\forall t \geq 0, \forall i = 1, \dots, n$, the solution $\mathcal{R}(t)$ of system (7) is defined on $[0, +\infty[$ and, by (12), has the property

$$\frac{d}{dt} (F_i(\mathcal{R}(t))) = -k F_i(\mathcal{R}(t)), \quad \forall t \geq 0, \forall i = 1, \dots, n$$

which implies the thesis. ■

Remark 2: As in remark 1, the control functions $v_F(\xi, \omega_0, v_0) = (v_1, v_2, \dots, v_n)^T$, $\omega_F(\xi, \omega_0, v_0) = (\omega_1, \omega_2, \dots, \omega_n)^T$ appearing in (13) can be computed iteratively in the following way:

$$\begin{aligned} \begin{pmatrix} v_1 \\ \omega_1 \end{pmatrix} &= -A_{11}^{-1}(\xi) \left(A_{10}(\xi) \begin{pmatrix} v_0 \\ \omega_0 \end{pmatrix} - k F_1(\xi_0, \xi_1) \right) \\ \begin{pmatrix} v_2 \\ \omega_2 \end{pmatrix} &= -A_{22}^{-1}(\xi) \left(A_{21}(\xi) \begin{pmatrix} v_1 \\ \omega_1 \end{pmatrix} + A_{20}(\xi) \begin{pmatrix} v_0 \\ \omega_0 \end{pmatrix} - k F_2(\xi_0, \xi_1, \xi_2) \right) \\ &\vdots \\ \begin{pmatrix} v_n \\ \omega_n \end{pmatrix} &= -A_{nn}^{-1}(\xi) \left(\sum_{j=0}^{n-1} A_{ij}(\xi) \begin{pmatrix} v_j \\ \omega_j \end{pmatrix} - k F_n(\xi) \right). \end{aligned} \quad (14)$$

In (12), the two terms appearing on the right-hand side have the following meaning: the term $-\sum_{j=0}^{i-1} A_{ij}(\xi) \begin{pmatrix} v_j \\ \omega_j \end{pmatrix}$ is the one which makes the constraint set $\mathcal{F}$ a controlled invariant, while $-k F_i(\xi)$ is the stabilizing term, which is null on the constraint set $\mathcal{F}$.

Remark 3: Note that in order to achieve finite-time convergence to the constraint set $\mathcal{F}$ starting from any $\xi \in \mathcal{F}(\epsilon)$, it is sufficient to consider the solution of system (7) with the controls defined by $\forall i = 1, \dots, n$

$$\begin{pmatrix} v_i \\ \omega_i \end{pmatrix} = A_{ii}^{-1}(\xi) \left(- \sum_{j=0}^{i-1} A_{ij}(\xi) \begin{pmatrix} v_j \\ \omega_j \end{pmatrix} - k \sigma(F_i(\xi)) \right) \quad (15)$$

where, for any vector $w = (w_1, w_2)^T \in \mathbb{R}^2$ we set

$$\sigma(w) = \left(\sqrt{|w_1|} \text{sign } w_1, \sqrt{|w_2|} \text{sign } w_2 \right)^T.$$

Then, $\forall i = 1, \dots, n$ we have that

$$\frac{d}{dt} (F_i(\mathcal{R}(t))) = -k \sigma(F_i(\mathcal{R}(t))), \quad \forall t \geq 0, \forall i = 1, \dots, n$$

which implies that $\forall i = 1, \dots, n, h = 1, 2$

$$|F_{ih}(\mathcal{R}(t))| = \begin{cases} \frac{(2\sqrt{|F_{ih}(\bar{\xi})| - kt})^2}{4}, & \text{if } t \leq \frac{2\sqrt{|F_{ih}(\bar{\xi})|}}{k} \\ 0, & \text{if } t > \frac{2\sqrt{|F_{ih}(\bar{\xi})|}}{k} \end{cases}$$

being $\forall i = 1, \dots, n, F_i(\xi) = (F_{i1}(\xi), F_{i2}(\xi))^T$. Therefore the solution reaches the constraint set within the time

$$\bar{t} = \max \left\{ \frac{2\sqrt{|F_{ih}(\bar{\xi})|}}{k}, i = 1, \dots, n, h = 1, 2 \right\}$$

since $F_i(\mathcal{R}(t)) = 0, \forall t \geq \bar{t}, \forall i = 1, \dots, n$.

Definition 5: Let F be a constraint function. We say that F is **regular** if property (6) holds.

Remark 4: A directed graph (V, A) can be naturally associated to a regular constraint function F in the following way. The node set $V = \{r_0, r_1, \dots, r_n\}$ represents the $(n+1)$ -tuple of robots in F -formation. The set A of edges contains all ordered pairs $(r_i, r_j) \in V \times V, i = 1, \dots, n, j = 0, \dots, n, i \neq 0$, such that $\partial_{\xi_j} F_i(\xi) \neq 0$. By hypotheses (3) and (6) the graph (V, A) is acyclic and weakly connected.

Remark 5: Suppose that F is an $(n+1)$ -tuple of robots in F -formation. Then, for any $\lambda \in C^1([0, +\infty[, [0, +\infty[), \tilde{\mathcal{R}} = \mathcal{R} \circ \lambda$ is still an $(n+1)$ -tuple of robots and $\tilde{\mathcal{R}}$ is in F -formation. Hence, if $v_0(t) \neq 0, \forall t \geq 0$, taking $\lambda(t) = \int_0^t v_0^{-1}(\tau) d\tau$, the velocity of the leader of $\tilde{\mathcal{R}}$ is always 1.

The previous remark justifies the following assumption:

Assumption 1: We henceforth suppose that the velocity v_0 of the leader r_0 is 1. As a consequence, ω_0 represents the curvature of the path (x_0, y_0) followed by the leader r_0 .

To properly study the relative position and orientation of the robots, we introduce the following definition. In the following, with rototranslation we mean a rigid-body motion obtained by composing a translation and a rotation.

Definition 6: For any parameter $p = (\bar{x}, \bar{y}, \bar{\theta})^T \in \mathbb{R}^2 \times S^1$, define the map $h_p : \mathbb{R}^2 \times S^1 \rightarrow \mathbb{R}^2 \times S^1$ as

$$h_p(x, y, \theta) = \left(R(\bar{\theta}) \begin{pmatrix} x + \bar{x} \\ y + \bar{y} \end{pmatrix}, \theta + \bar{\theta} \right) \quad (16)$$

where $R(\theta) = (\tau(\theta), \eta(\theta))$. For any $\xi = (\xi_0, \xi_1, \dots, \xi_n) \in \mathcal{X}$, set $H_p(\xi) = (h_p(\xi_0), h_p(\xi_1), \dots, h_p(\xi_n))$. A map $G \in C^1(\mathcal{X}, \mathbb{R}^\ell)$ is called **rototranslation invariant** if

$$G(\xi) = G(H_p(\xi)), \forall \xi \in \mathcal{X}, \forall p \in \mathbb{R}^2 \times S^1.$$

In other words, a function of the configuration space $\mathcal{X}$ is rototranslation invariant when its value does not change when any rototranslation H_p is applied to its argument (see Fig. 4). Let us now introduce the following equivalence relation $\sim$ on $\mathcal{X}$.

Definition 7: Given two vectors $\xi, \eta \in \mathcal{X}$

$$\eta \sim \xi \text{ if } \exists p \in \mathbb{R}^2 \times S^1 \text{ such that } \eta = H_p(\xi).$$

We denote by $[\xi] = \{\eta \in \mathcal{X} | \eta \sim \xi\}$ the equivalence class of ξ and by $\mathcal{X} / \sim = \{[\xi] | \xi \in \mathcal{X}\}$ the quotient set.

Fig. 4. If the constraint function F is rototranslation invariant, the robots in (a) are in F -formation if and only if the robots in (b) are in F -formation, since the team in (a) can be superimposed to the team in (b) via a rototranslation.

Note that G is a rototranslation invariant function if and only if for any $\xi \in \mathcal{X}$, G is constant on the set $[\xi]$.

In the case of rototranslation invariant constraint functions, it is natural to define a *reduced constraint set* as follows.

Definition 8: The **reduced constraint set** $\mathcal{F}_r$ is the set $\mathcal{F}_r = \mathcal{F} / \sim$.

In this way, each element of $\mathcal{F}_r$ represents a set of configurations for the formation that differ only by a rototranslation.

Given an $(n+1)$ -tuple of robots $\mathcal{R}$, the **reduced motion** of $\mathcal{R}$ is the map $[\mathcal{R}] : [0, +\infty[\rightarrow \mathcal{X} / \sim$ defined by $[\mathcal{R}](t) = [\mathcal{R}(t)], \forall t \geq 0$. Note that $[\mathcal{R}]$ describes the motion of the followers in the leader's reference frame.

The following proposition presents some properties of the reduced motion of an $(n+1)$ -tuple of robots.

Proposition 3: Let F be a rototranslation invariant function.

- 1) Let $\mathcal{R} = (r_0, \dots, r_n)$ be an $(n+1)$ -tuple of robots. The reduced motion $[\mathcal{R}]$ of $\mathcal{R}$ is constant if and only if there exists $\bar{\xi} \in \mathcal{F}$ such that

$$H_{-r_0(t)}(\mathcal{R}(t)) = \bar{\xi}, \quad \forall t \geq 0. \quad (17)$$

This means that the position and the orientation of the followers, with respect to the leader's reference frame, is fixed.

- 2) Let $\mathcal{R}_1, \mathcal{R}_2$ be two $(n+1)$ -tuples of robots in F -formation, with the same controls v_0 and ω_0 for the leader. Suppose that

$$\mathcal{R}_1(0) \sim \mathcal{R}_2(0) \quad (18)$$

then

$$\mathcal{R}_1(t) \sim \mathcal{R}_2(t), \quad \forall t \geq 0 \quad (19)$$

that is, if $[\mathcal{R}_1(0)] = [\mathcal{R}_2(0)]$, then

$$[\mathcal{R}_1(t)] = [\mathcal{R}_2(t)], \quad \forall t \geq 0.$$

Proof: Part 1) of the statement is evident. For part 2), by (18) there exists $p \in \mathbb{R}^2 \times S^1$, such that

$$H_p(\mathcal{R}_1(0)) = \mathcal{R}_2(0).$$

Proving (19) is equivalent to show that

$$H_p(\mathcal{R}_1(t)) = \mathcal{R}_2(t), \quad \forall t \geq 0. \quad (20)$$

In fact $F(H_p(\mathcal{R}_1(t))) = 0$ since F is rototranslation invariant, moreover $H_p(\mathcal{R}_1(t))$ is still an $(n+1)$ -tuple of robots. Then (20) holds by 3) of proposition 1. ■

Remark 6: Let F be a rototranslation invariant constraint function. Set

$$\Gamma_F = \{\xi = (\xi_0, \dots, \xi_n)^T \in \mathcal{X} | \xi_0 = 0, F(\xi) = 0\}. \quad (21)$$

Then the map $\mathcal{J} : \mathcal{F}_r \longrightarrow \Gamma_F$ defined by

$$\mathcal{J}(\llbracket \xi \rrbracket) = H_{-\xi_0}(\xi) \quad (22)$$

is a bijection. Moreover an $(n+1)$ -tuple of robots $\mathcal{R}$ is in F -formation if and only if $H_{-\mathbf{r}_0(t)}(\mathcal{R}(t)) \in \Gamma_F, \forall t \geq 0$.

As suggested by (22), $\mathcal{F}_r$ is the set of all configurations of the followers in the leader's reference frame, that are compatible with the constraint function.

The following proposition states that if F is a rototranslation invariant constraint function, then each follower can compute its control input by only knowing the pose of the preceding robots in its own reference frame. In this way, the control law of the followers can be implemented by relying on the information provided by on-board sensors.

Proposition 4: If F is rototranslation invariant, then for any $i = 1, \dots, n, j = 0, \dots, n$ and for any $p \in \mathbb{R}^2 \times S^1$

$$A_{ij}(\xi) = A_{ij}(H_p(\xi)), \quad \forall \xi \in \mathcal{X} \quad (23)$$

i.e., terms A_{ij} are constant on the equivalence classes of $X/\sim$. Therefore the controls v_i, ω_i given by (10), (13), and (15) do not change by replacing ξ with $H_{-\xi_i}(\xi)$, i.e., they can be expressed just in terms of the coordinates relative to the reference frame of the i th robot.

Proof: Set $p = (\bar{x}, \bar{y}, \bar{\theta})^T \in \mathbb{R}^2 \times S^1, \forall j = 0, \dots, n$. Since F is rototranslation invariant, $\forall i = 1, \dots, n, \forall \eta \in (\mathbb{R}^2 \times S^1)^n$

$$\begin{aligned} F'_i(\xi)\eta &= \lim_{h \rightarrow 0} \frac{F_i(\xi + h\eta) - F_i(\xi)}{h} \\ &= \lim_{h \rightarrow 0} \frac{F_i(H_p(\xi + h\eta)) - F_i(H_p(\xi))}{h} \\ &= \lim_{h \rightarrow 0} \frac{F_i(H_p(\xi) + hH'_p(\xi)\eta) - F_i(H_p(\xi))}{h} \\ &= F'_i(H_p(\xi)) H'_p(\xi)\eta \end{aligned} \quad (24)$$

where $H'_p(\xi)$ is the block-diagonal matrix with $(n+1)$ blocks given by the constant 3×3 matrix $\begin{pmatrix} R(\bar{\theta}) & (0,0)^T \\ (0,0) & 1 \end{pmatrix}$. For any $j = 0, \dots, n$, set

$$\begin{aligned} \eta_j &= (0, \dots, (0,0,1)_j, 0, \dots, 0)^T \in (\mathbb{R}^2 \times S^1)^{n+1}, \\ \zeta_j &= (0, \dots, (\cos \bar{\theta}, \sin \bar{\theta})_j, 0, \dots, 0)^T \in (\mathbb{R}^2 \times S^1)^{n+1} \end{aligned}$$

then by (24)

$$\begin{aligned} A_{ij}(\xi) &= (F'_i(\xi)\zeta_j, F'_i(\xi)\eta_j) \\ &= (F'_i(H_p(\xi)) H'_p(\xi)\zeta_j, F'_i(H_p(\xi)) H'_p(\xi)\eta_j) \\ &= (\cos(\theta_j + \bar{\theta}) \partial_{x_j} F_i(H_p(\xi)) \\ &\quad + \sin(\theta_j + \bar{\theta}) \partial_{y_j} F_i(H_p(\xi)), \partial_{\theta_j} F_i(H_p(\xi))) \\ &= A_{ij}(H_p(\xi)). \end{aligned}$$

■

Since $\mathcal{F}_r$ is the quotient set of $\mathcal{F}$ by the equivalence relation $\sim$, it has a lower dimension than $\mathcal{F}$ as specified in the following proposition.

Proposition 5: If F is a regular rototranslation invariant constraint function, $\mathcal{F}_r$ is a differentiable manifold of dimension n , diffeomorphic to Γ_F .

Proof: By (6) and the implicit function theorem, the subset of $\mathcal{F}$ given by Γ_F [see (21)] is a submanifold of dimension n of $\mathcal{X}$. Since the map (22) is a bijection, this induces a natural structure of differentiable manifold on $\mathcal{F}_r$. ■

In the following definition, ρ_F represents the minimum radius of a circle centered at the leader, which contains all the positions that the followers can assume while respecting the constraint F .

Definition 9: For any $i = 1, \dots, n$, let $\Pi_i : \mathcal{X} \rightarrow \mathbb{R}^2$ be the map defined by $\Pi_i(\xi) = (x_i, y_i)^T$ and let F be a rototranslation invariant constraint function. Set

$$\rho_F = \sup \{ \|\Pi_i(\xi)\| | \xi \in \Gamma_F \} \quad (25)$$

which is the radius of the smallest circle centered at $(0,0)^T$ and containing all $(x_i, y_i)^T = \Pi_i(\xi)$, for any $i = 1, \dots, n$, and all possible configuration $\xi \in \Gamma_F$, that is

$$\bigcup_{i=1}^n \Pi_i(\Gamma_F) \subset B((0,0)^T, \rho_F).$$

Proposition 6: Let F be regular and rototranslation invariant. If $\rho_F < +\infty$, then there exists $\epsilon > 0$ such that, $\forall \xi \in \mathcal{F}(\epsilon), \forall i = 1, \dots, n$

$$\det(\cos \theta_i \partial_{x_i} F_i(\xi) + \sin \theta_i \partial_{y_i} F_i(\xi), \partial_{\theta_i} F_i(\xi)) \neq 0$$

i.e., hypothesis (11) of proposition 2 is satisfied.

Proof: Set Γ_F is compact since it is a closed subset of $(B((0,0)^T, \rho_F) \times S^1)^n$ by the definition of ρ_F . Let $\Phi : \mathcal{X} \rightarrow (\mathbb{R}^2 \times S^1) \times \mathbb{R}^{2n}$ be the map defined by $\Phi(\xi) = (\xi_0, F(\xi))$, $\forall \xi \in \mathcal{X}$. Since $\Gamma_F = \{\xi \in \mathcal{X} | \Phi(\xi) = 0\}$, condition (6) holds $\forall \xi \in \Gamma_F$ and Γ_F is compact, then there exists an $\epsilon > 0$ such that, $\forall \xi \in \Phi^{-1}(B(0, \epsilon))$

$$\det(\cos \theta_i \partial_{x_i} F_i(\xi) + \sin \theta_i \partial_{y_i} F_i(\xi), \partial_{\theta_i} F_i(\xi)) \neq 0 \quad (26)$$

where $\Phi^{-1}(B(0, \epsilon))$ is the counter image of $B(0, \epsilon) = \{\eta \in \mathbb{R}^{2n+3} | \|\eta\| \leq \epsilon\}$. Since $\mathcal{F}(\epsilon) = \{\xi \in \mathcal{X} | \|F(\xi)\| \leq \epsilon\}$, $\Phi^{-1}(B(0, \epsilon)) = \{H_{-\xi_0}(\xi) | \xi \in \mathcal{F}(\epsilon)\}$, that is $\Phi^{-1}(B(0, \epsilon))$ is a set of representative elements of the equivalence classes $\llbracket \xi \rrbracket$ with $\xi \in \mathcal{F}(\epsilon)$. Moreover, by (23), $\det(\cos \theta_i \partial_{x_i} F_i(\xi) + \sin \theta_i \partial_{y_i} F_i(\xi), \partial_{\theta_i} F_i(\xi))$ is constant on the elements ξ of the same equivalence class, since F is rototranslation invariant, therefore, $\forall \xi \in \mathcal{F}(\epsilon)$

$$\det(\cos \theta_i \partial_{x_i} F_i(\xi) + \sin \theta_i \partial_{y_i} F_i(\xi), \partial_{\theta_i} F_i(\xi)) \neq 0$$

by (26). ■

Let F be a regular rototranslation invariant constraint function and let $\omega_0 \in \mathcal{C}([0, +\infty[, \mathbb{R})$ be the control for the leader. A consequence of part 2) of proposition 3 is that the following map is well defined:

$$\begin{aligned} \Phi_{\omega_0} : [0, +\infty[\times \mathcal{F}_r &\longrightarrow \mathcal{F}_r \\ (t, \llbracket \xi \rrbracket) &\rightsquigarrow \Phi_{\omega_0}(t, \llbracket \xi \rrbracket) \end{aligned}$$

Fig. 5. For a three-robot formation, condition (30) prevents the robots r_0, r_1, r_2 from being all with the same orientation and aligned on a straight line orthogonal to their heading direction.

where

$$\Phi_{\omega_0}(t, \llbracket \zeta \rrbracket) = \llbracket \xi(t) \rrbracket \quad (27)$$

being ξ the unique solution of the following system:

$$\begin{cases} \dot{\xi} = g(\xi, 1, \omega_0, v_F(\xi, 1, \omega_0), \omega_F(\xi, 1, \omega_0)) \\ \xi(0) = \zeta. \end{cases}$$

where g is given by (5) and v_F, ω_F by proposition 1.

Definition 10: Set $\forall \omega_0 \in \mathbb{R}, \forall \llbracket \xi \rrbracket \in \mathcal{F}_r$.

$$\Psi(\llbracket \xi \rrbracket, \omega_0) = \frac{d}{dt}(\Phi_{\omega_0}(t, \llbracket \xi \rrbracket))|_{t=0} \in T_{\mathcal{F}_r}(\llbracket \xi \rrbracket)$$

where $T_{\mathcal{F}_r}(\llbracket \xi \rrbracket)$ is the tangent space of $\mathcal{F}_r$ at $\llbracket \xi \rrbracket$. Then $\forall \omega_0 \in \mathcal{C}([0, +\infty[, \mathbb{R}), \forall \llbracket \zeta \rrbracket \in \mathcal{F}_r$, $\Phi_{\omega_0}(t, \llbracket \zeta \rrbracket)$ is the solution of the following system defined on the manifold $\mathcal{F}_r$:

$$\begin{cases} \dot{\Phi}(t) = \Psi(\Phi(t), \omega_0(t)) \\ \Phi(0) = \llbracket \zeta \rrbracket \end{cases} \quad (28)$$

which we call the **reduced internal dynamics** of the formation. Thanks to the diffeomorphism between $\mathcal{F}_r$ and Γ_F , it represents the motion of the followers in the leader's reference frame.

As a consequence of (28), given $\llbracket \xi \rrbracket \in \mathcal{F}_r$ and $\omega_0 \in \mathbb{R}$, $(\llbracket \xi \rrbracket, \omega_0)$ is an equilibrium point of Ψ , that is $\Psi(\llbracket \xi \rrbracket, \omega_0) = 0$ if and only if the reduced motion $\llbracket \mathcal{R} \rrbracket$ of the $(n+1)$ -tuple of robots $\mathcal{R}$ (which is in F -formation and $\mathcal{R}(0) = \xi$) is constant. Roughly speaking, the equilibria of Ψ are the configurations in which the position and orientation of the followers are constant in the leader's reference frame.

The following theorem characterizes the $(n+1)$ -tuple of robots $\mathcal{R}$ whose reduced motion $\llbracket \mathcal{R} \rrbracket$ is constant. In particular, it shows that if condition (30) is satisfied, then the position and the orientation of each follower with respect to the leader's reference frame are fixed (i.e., the followers are stationary in the leader's frame) if and only if the leader and the followers either move along circular paths or parallel straight lines.

The nonalignment condition (30) means that the manifold $\mathcal{F}$ must not contain configurations in which all the followers are placed on a straight line passing through the leader and orthogonal to its heading direction $\tau(\theta_0)$, with the followers oriented as the leader (see Fig. 5).

Theorem 1: Suppose that F is a regular rototranslation invariant constraint function and that $\mathcal{R}$ is an $(n+1)$ -tuple of robots in F -formation. Let us consider the following properties:

1) ω_0 is constant and

a) if $\omega_0 = 0$, then $\theta_0(t) = \bar{\theta}_0, \forall t \geq 0$ and, $\forall i = 1, \dots, n$

$$\theta_i(t) \in \{\bar{\theta}_0, \bar{\theta}_0 + \pi\}, \quad \forall t \geq 0.$$

b) if $\omega_0 \neq 0$, then there exists $(\bar{x}, \bar{y})^T \in \mathbb{R}^2$ such that $\forall t \geq 0, \forall i = 0, \dots, n$

$$\left\langle \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \tau(\theta_i(t)) \right\rangle = 0. \quad (29)$$

2) $\llbracket \mathcal{R} \rrbracket$ is constant.

Then 1) implies 2). Vice versa, if the following property holds:

$$\begin{cases} \mathcal{F} \text{ does not contain any point of the set :} \\ \Sigma = \left\{ \xi \in \mathcal{X} \mid \left\langle \begin{pmatrix} x_i - x_0 \\ y_i - y_0 \end{pmatrix}, \tau(\theta_0) \right\rangle = 0, \right. \\ \left. \theta_i \in \{\theta_0, \theta_0 + \pi\}, \forall i = 1, \dots, n \right\}. \end{cases} \quad (30)$$

then 2) implies 1).

The proof is reported in the Appendix.

Definition 11: For any $i = 1, \dots, n$, let $\Theta : \mathcal{X} \rightarrow S^1$ be the map defined by

$$\Theta_i(\xi) = \theta_i$$

(recall that $\xi = (\xi_0, \xi_1, \dots, \xi_n)^T$ and $\xi_i = (x_i, y_i, \theta_i)^T, \forall i = 1, \dots, n$).

Therefore, if we set the n -dimensional torus $T^n = \{\beta = (\beta_1, \dots, \beta_n)^T \mid \beta_i \in S^1, i = 1, \dots, n\}$, the map $\Theta : \mathcal{X} \rightarrow T^n$ defined by $\Theta(\xi) = (\Theta_1(\xi), \dots, \Theta_n(\xi))^T = (\theta_1, \dots, \theta_n)^T$, is the projection of $\mathcal{X}$ on the torus of the followers' heading angles.

Suppose that F is a regular rototranslation invariant constraint function such that, $\forall \xi \in \Gamma_F$

$$\det(\partial_{x_1} F(\xi), \partial_{y_1} F(\xi), \dots, \partial_{x_n} F(\xi), \partial_{y_n} F(\xi)) \neq 0. \quad (31)$$

By the implicit function theorem, the restriction on Γ_F of the projection

$$\begin{aligned} \Theta : \Gamma_F &\longrightarrow T^n \\ \xi &\rightsquigarrow \Theta(\xi) = (\theta_1, \dots, \theta_n)^T \end{aligned}$$

is a local $\mathcal{C}^1$ diffeomorphism. Observe that, since $\xi \in \Gamma_F$, θ_i represents the relative heading angle between the i th robot and the leader. This justifies the following definition.

Definition 12: The function F is called a **relative angle parametrizable constraint function** if it satisfies the following properties:

- 1) F is regular;
- 2) F is rototranslation invariant;
- 3) F verifies property (31).

Remark 7: Suppose that F is a relative angle parametrizable constraint function. For any $\omega_0 \in \mathcal{C}([0, +\infty[, \mathbb{R})$ set $\varphi_{\omega_0} : [0, +\infty[\times \Gamma_F \rightarrow \Gamma_F$ the map defined by $\varphi_{\omega_0}(t, \xi) = \mathcal{J}(\Phi_{\omega_0}(t, \llbracket \xi \rrbracket))$, where $\mathcal{J}$ is given by (22). For any $\bar{\xi} \in \Gamma_F$, set $\bar{\beta} = \Theta(\bar{\xi}) \in T^n$. Since Θ is a local diffeomorphism, there exists a neighborhood U of $\bar{\xi} \in \Gamma_F$ on which Θ is a diffeomorphism. Then the reduced internal dynamics system (28) becomes (in the torus variables of $\Theta(U)$)

$$\begin{cases} \dot{\beta} = \psi(\beta(t), \omega_0(t)) \\ \beta(0) = \bar{\beta} \end{cases} \quad (32)$$

Fig. 8. (a) Sample formation of four robots, with $\lambda_{i,k} = 1/i$, $i = 1, 2, 3$, $k = 0, \dots, i-1$. (b) Formation belonging to the class studied in [40].

given by (69) (see the Appendix). A root-finding algorithm for a scalar function, for example the bisection method, can be used for this purpose.

III. APPLICATION TO A SPECIFIC HIERARCHICAL CONSTRAINT FUNCTION

In this section, we apply the theoretical results of Section II to a specific constraint function F . To define $F = (F_1, \dots, F_n)$, we assign for any $i = 1, \dots, n$ the following parameters:

$$\lambda_{i,0}, \dots, \lambda_{i,i-1} \geq 0 : \sum_{k=0}^{i-1} \lambda_{i,k} = 1, d_i > 0, -\frac{\pi}{2} < \phi_i < \frac{\pi}{2}$$

and set

$$F_i(\xi_0, \xi_1, \dots, \xi_i) = \sum_{k=0}^{i-1} \lambda_{i,k} \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \begin{pmatrix} x_i \\ y_i \end{pmatrix} - d_i \tau(\theta_i + \phi_i). \quad (38)$$

With this constraint function, we require robot i to follow an assigned convex combination of the positions of the preceding $i-1$ robots, at a fixed distance d_i and with a fixed visual angle ϕ_i [see Fig. 8(a)].

This particular function may be useful to describe formations occurring in nature, such as, e.g., birds flocks, where it is believed that each bird follows an average of the preceding ones [41].

Note that if, $\forall i = 1, \dots, n$, there exists an integer l_i such that $0 \leq l_i \leq i-1$ and $\lambda_{i,k} = \begin{cases} 0, & \text{if } k \neq l_i \\ 1, & \text{if } k = l_i \end{cases}$ then F induces the kind of “hierarchical” formations studied in [40] [see Fig. 8(b)]. In addition, if $i = 1$, we end up with the leader-follower formations investigated in [39].

F is a regular constraint function since (3) holds and (6) is satisfied. In fact $|\phi_i| < \pi/2$ by hypothesis and $\det(\cos \theta_i \partial_{x_i} F_i + \sin \theta_i \partial_{y_i} F_i, \partial_{\theta_i} F_i) = d_i \cos \phi_i$. Clearly, F is rototranslation invariant, and it satisfies (30) and (31). Therefore, F is a relative angle parameterizable constraint function which verifies the nonalignment condition (30) and the results developed in Section II can be applied. In particular, from proposition 1, we

have that if $\mathcal{R}$ is an $(n+1)$ -tuple of robots, which is in F -formation at the initial time (i.e., $F(\mathcal{R}(0)) = 0$), then for any trajectory of the leader (i.e., $\forall \omega_0 \in C^0([0, +\infty[, \mathbb{R})$, recall that v_0 is assumed to be equal to 1), there exist and are unique the controls $v_F = (v_1, \dots, v_n)^T$, $\omega_F = (\omega_1, \dots, \omega_n)^T$ for the followers such that $\mathcal{R}$ is in F -formation for any $t \geq 0$ (i.e., $F(\mathcal{R}(t)) = 0$, $\forall t \geq 0$). These controls are given by

$$\begin{aligned} v_i(t) &= \frac{1}{\cos \phi_i} \sum_{h=0}^{i-1} \lambda_{i,h} v_h(t) \cos(\theta_h(t) - \theta_i(t) - \phi_i) \\ \omega_i(t) &= \frac{1}{d_i \cos \phi_i} \sum_{h=0}^{i-1} \lambda_{i,h} v_h(t) \sin(\theta_h(t) - \theta_i(t)) \end{aligned} \quad (39)$$

and the stabilizing controls [see (14)] are given by

$$\begin{aligned} v_i(t) &= \frac{1}{\cos \phi_i} \left(\sum_{h=0}^{i-1} \lambda_{i,h} v_h(t) \cos(\theta_h(t) - \theta_i(t) - \phi_i) \right. \\ &\quad \left. - k\tau(\theta_i(t) + \phi_i)^T \right. \\ &\quad \left. \times \left(\sum_{h=0}^{i-1} \lambda_{i,h} \begin{pmatrix} x_h(t) \\ y_h(t) \end{pmatrix} - \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} \right) \right. \\ &\quad \left. - d_i \tau(\theta_i(t) + \phi_i) \right), \\ \omega_i(t) &= \frac{1}{d_i \cos \phi_i} \left(\sum_{h=0}^{i-1} \lambda_{i,h} v_h(t) \sin(\theta_h(t) - \theta_i(t)) \right. \\ &\quad \left. - k\nu(\theta_i(t))^T \right. \\ &\quad \left. \times \left(\sum_{h=0}^{i-1} \lambda_{i,h} \begin{pmatrix} x_h(t) \\ y_h(t) \end{pmatrix} - \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} \right) \right. \\ &\quad \left. - d_i \tau(\theta_i(t) + \phi_i) \right) \end{aligned}$$

where k is a positive real number. In particular, from theorem 1 we have that an $(n+1)$ -tuple of robots $\mathcal{R}$ in F -formation has a fixed configuration in the leader's reference frame (i.e., $\|\mathcal{R}(t)\|$ is constant) if and only if all robots move along straight lines or concentric circles. From theorem 2 we have that *such a fixed relative configuration exists when the leader's curvature ω_0 is constant and $|\omega_0| < 1/\rho_F$, where ρ_F is given by definition 9. In this specific example, ρ_F can be bounded as follows:*

$$\rho_F \leq \max_{i=1, \dots, n} \bar{\rho}_i \quad (40)$$

where $\bar{\rho}_0 = 0$ and $\bar{\rho}_i = d_i + \sum_{k=0}^{i-1} \lambda_{i,k} \bar{\rho}_k$. Moreover, the i th component of the reduced internal dynamics system (32) is given by

$$\dot{\beta}_i(t) = \psi_i(\beta(t), \omega_0(t)), \quad i = 1, \dots, n \quad (41)$$

where

$$\begin{aligned} \psi_i(\beta(t), \omega_0(t)) &= -\omega_0(t) \\ &\quad - \frac{1}{d_i \cos \phi_i} \sum_{k=0}^{i-1} \lambda_{i,k} v_k(\beta) \sin(\beta_i(t) - \beta_k(t)) \end{aligned} \quad (42)$$

being

$$v_i(\beta) = \frac{1}{\cos \phi_i} \sum_{k=0}^{i-1} \lambda_{i,k} v_k(\beta) \cos(\beta_k - \beta_i - \phi_i). \quad (43)$$

These expressions are obtained by computing the derivatives of $\beta_i = \theta_i - \theta_0$ and taking into account (39) (notice that β_i represents the relative heading angle between the i th robot and the leader). Note that, in this case, the projection $\Theta : \Gamma_F \rightarrow T^n$ is a global diffeomorphism, not only local as in the general case (see remark 7).

Remark 9: There is a strong resemblance between (42) and Kuramoto model for coupled oscillators (see for instance equation (3.1) of [33]), which has the form

$$\dot{\theta}_i = \omega_i + \frac{K}{n} \sum_{k=1}^n \sin(\theta_k - \theta_i), \quad i = 1, \dots, n$$

where θ_i denotes the phase of oscillator i , ω_i its natural frequency and $K \geq 0$ is the coupling strength.

In fact, (42) can be rewritten in terms of $\theta = (\theta_1, \dots, \theta_n)^T$ in the following way:

$$\dot{\theta}_i = \frac{1}{d_i \cos \phi_i} \sum_{k=0}^{i-1} \lambda_{i,k} v_k(\theta) \sin(\theta_k - \theta_i), \quad i = 1, \dots, n \quad (44)$$

where $v_k(\theta)$ is given by (43).

The similarity is evident, note however the following two differences.

- 1) In (44) the leader's heading angle θ_0 is an assigned function. As it will be shown in theorem 3, the angles θ_i , solutions of (44), follow θ_0 by maintaining themselves at a maximum distance from it that depends on the maximum value of θ_0 (see property (48), below).
- 2) In (44), the sine function is multiplied by the velocity $v_k(\theta)$. Hence, in our case, the coupling strength K is a nonlinear function of the oscillators' state.

Given an $(n+1)$ -tuple of robots, which is in F -formation, the following theorem provides a method to determine an invariant region for the reduced internal dynamics (41), whose size depends on the bounds on the curvature of the path followed by the leader. In other words, this theorem allows to find a bounded region in the leader's reference frame, where the followers are confined when the team moves.

Theorem 3: Set $v_0 = 1$, let $\omega_0^- \leq \omega_0^+$ be real constants and define the following constants:

$$\beta_0^- = \beta_0^+ = 0$$

$\forall i = 1, \dots, n$

$$\begin{aligned} \beta_i^+ = \max \left\{ \arg \left(\sum_{k=0}^{i-1} \lambda_{i,k} z_k e^{\beta_k^+ j} \right) \right. \\ \left. - \arcsin \left(\frac{\omega_0^+ d_i \cos \phi_i}{\left| \sum_{k=0}^{i-1} \lambda_{i,k} z_k e^{\beta_k^+ j} \right|} \right) \right\} \\ |(z_0, \dots, z_{i-1}) \in V_i \end{aligned}$$

$$\begin{aligned} \beta_i^- = \min \left\{ \arg \left(\sum_{k=0}^{i-1} \lambda_{i,k} z_k e^{\beta_k^- j} \right) \right. \\ \left. - \arcsin \left(\frac{\omega_0^- d_i \cos \phi_i}{\left| \sum_{k=0}^{i-1} \lambda_{i,k} z_k e^{\beta_k^- j} \right|} \right) \right\} \\ |(z_0, \dots, z_{i-1}) \in V_i \end{aligned} \quad (45)$$

where $e^{\alpha j} = \cos \alpha + j \sin \alpha$, $\forall \alpha \in S^1$ and $V_i = \{1\} \times \{v_1^-, v_1^+\} \times \dots \times \{v_{i-1}^-, v_{i-1}^+\}$

$$v_0^- = v_0^+ = 1,$$

$$v_i^- = \frac{1}{\cos \phi_i} \sum_{k=0}^{i-1} \lambda_{i,k} v_k^- \cos(\max\{|\phi_i - (\beta_k^+ - \beta_i^-)|, |\phi_i - (\beta_k^- - \beta_i^+)|\})$$

$$v_i^+ = \frac{1}{\cos \phi_i} \sum_{k=0}^{i-1} \lambda_{i,k} v_k^+ \cos a_{ik}$$

where

$$a_{ik} = \begin{cases} 0, & \text{if } \phi_i - (\beta_k^+ - \beta_i^-) \leq 0 \leq \phi_i - (\beta_k^- - \beta_i^+) \\ \min\{|\phi_i - (\beta_k^+ - \beta_i^-)|, |\phi_i - (\beta_k^- - \beta_i^+)|\}, & \text{otherwise.} \end{cases}$$

Set $\mathcal{B} = [\beta_1^-, \beta_1^+] \times \dots \times [\beta_n^-, \beta_n^+]$. If

$$v_i^- \geq 0, |\beta_i^\pm| < \frac{\pi}{4}, \quad \forall i = 1, \dots, n \quad (46)$$

and $\omega_0 \in C^0([0, +\infty[, \mathbb{R})$ is such that

$$\omega_0^- \leq \omega_0(t) \leq \omega_0^+ \quad (47)$$

then the following invariance property holds:

If β is the solution of (41) with $\beta(0) \in \mathcal{B}$, then

$$\beta(t) \in \mathcal{B}, \quad \forall t \geq 0 \quad (48)$$

and, $\forall i = 1, \dots, n$, the functions v_i given by (43), are such that

$$v_i^- \leq v_i(t) \leq v_i^+, \quad \forall t \geq 0. \quad (49)$$

Remark 10:

- 1) Property (48) guarantees that, during the motion, the relative orientation between each follower and the leader is bounded by constants $\beta_i^\pm$. This condition limits the possible variations of the formation shape in the leader's reference frame. Once the weights $\lambda_{i,k}$ have been assigned in function (38), one can compute $\beta_i^-, \beta_i^+, v_i^-, v_i^+$ for all values of ω_0^- and ω_0^+ for which (46) is satisfied. In this way it is possible to obtain bounds on the followers' velocities and on their misalignment with respect to the leader, for different choices of the maximum curvature allowed to the leader's trajectory.
- 2) There always exist (sufficiently small) values of positive and negative curvature for the path of the leader such that the hypotheses of theorem 3 are verified and condition (48) holds. In fact, since condition (46) of theorem 3 holds when $\omega_0^- = \omega_0^+ = 0$, by continuity, there exists a real constant

 Fig. 9. First simulation: (a) Trajectory of the four robots. (b) Time history of β_i .

$\epsilon > 0$ such that, if $\omega_0^- = -\epsilon$ and $\omega_0^+ = \epsilon$, condition (46) is still satisfied. Set

Proof of Theorem 3: Let $\omega_0 \in \mathcal{C}([0, +\infty[, \mathbb{R})$ be such that (48) holds and $\beta \in \mathcal{C}^1([0, +\infty[, \mathbb{R}^n)$ be a solution of the reduced internal dynamics system (41) such that $\beta(0) \in \mathcal{B}$. Note that (48) and (49) hold if we show that the following property is true (recall that $v_0(t) = 1, \forall t \geq 0$):

$$\left\{ \begin{array}{l} \text{Let } i \text{ be such that } 1 < i \leq n. \\ \text{If } \forall k = 0, \dots, i-1, \forall t \geq 0, \\ \quad \beta_k^- \leq \beta_k(t) \leq \beta_k^+, v_k^- \leq v_k(t) \leq v_k^+, \quad (50) \\ \text{then, } \forall t \geq 0 \\ \quad \beta_i^- \leq \beta_i(t) \leq \beta_i^+, \quad (51) \\ \quad v_i^- \leq v_i(t) \leq v_i^+. \quad (52) \end{array} \right.$$

Observe that, by (50)

$$(v_1(t), \dots, v_{i-1}(t))^T \in [v_1^-, v_1^+] \times \dots \times [v_{i-1}^-, v_{i-1}^+].$$

Note that V_i represents the set of the vertices of the polyhedron $\{1\} \times [v_i^-, v_i^+] \times \dots \times [v_{i-1}^-, v_{i-1}^+]$. The cardinality of V_i is 2^{i-1} and its generic element is denoted by $z = (z_0, \dots, z_{i-1})$. Then, if (50) is verified, there exist 2^{i-1} nonnegative continuous functions $\{\mu_z(t)\}_{z \in V_i}$ such that

$$\sum_{z \in V_i} \mu_z = 1 \text{ and } (1, v_1(t), \dots, v_{i-1}(t))^T = \sum_{z \in V_i} \mu_z(t) z.$$

Therefore,

$$\begin{aligned} \psi_i(\beta(t), \omega_0) &= -\omega_0(t) - \sum_{k=0}^{i-1} \lambda_{i,k} \left(\sum_{z \in V_i} \mu_z(t) z \right)_k \\ &\quad \times \sin(\beta_i(t) - \beta_k(t)) \\ &= \sum_{z \in V_i} \mu_z(t) \psi_{i,z}(\beta(t), \omega_0(t)) \end{aligned} \quad (53)$$

where $\forall z \in V_i$

$$\psi_{i,z}(\beta, \omega_0) = -\omega_0 - \sum_{k=0}^{i-1} \lambda_{i,k} z_k \sin(\beta_i - \beta_k).$$

$$\begin{aligned} S_+ &= [\beta_1^-, \beta_1^+] \times \dots \times [\beta_{i-1}^-, \beta_{i-1}^+] \times \{\beta_i^+\} \\ S_- &= [\beta_1^-, \beta_1^+] \times \dots \times [\beta_{i-1}^-, \beta_{i-1}^+] \times \{\beta_i^-\} \end{aligned}$$

then by definition (45) we get that $\forall z \in V_i$:

$$\begin{aligned} \psi_{i,z}(\beta, \omega_0) &\leq 0, \forall \beta \in S_+, \forall \omega_0 \in [\omega_0^-, \omega_0^+] \\ \psi_{i,z}(\beta, \omega_0) &\geq 0, \forall \beta \in S_-, \forall \omega_0 \in [\omega_0^-, \omega_0^+] \end{aligned}$$

which implies, by (53), that $\forall \omega_0 \in [\omega_0^-, \omega_0^+]$

$$\begin{aligned} \psi_i(\beta, \omega_0) &\leq 0, \forall \beta \in S_+ \\ \psi_i(\beta, \omega_0) &\geq 0, \forall \beta \in S_- \end{aligned}$$

and (51) holds. Finally (52) is true by direct computation. ■

IV. SIMULATION RESULTS

In the simulations described in this section, we have considered the constraint function (38) with $(d_1, \phi_1) = (d_2, \phi_2) = (d_3, \phi_3) = (1, \pi/4)$. We set $\lambda_{i,k} = 1/i, i = 1, 2, 3, k = 0, \dots, i-1$: in this way robot i follows exactly the average of the positions of the preceding $i-1$ robots in the formation. Applying (40), we have that $1/\rho_F \geq 0.546$. Let $\bar{\omega} = 0.25$, then, by theorem 2, there exists an equilibrium configuration $\bar{\beta}$ for which $\omega_0(t) = 0.25, \forall t \geq 0$. It can be obtained solving iteratively the fixed-point (36), or, equivalently, it is such that $\psi(\bar{\beta}, \bar{\omega}) = 0$, in (33). In this example, it corresponds to a solution of $\psi_i(\bar{\beta}, \bar{\omega}) = 0$ for $i = 1, 2, 3$, where ψ_i is reported in (42). An equilibrium configuration is given by $\bar{\beta} = (\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3)^T = (-0.177, -0.277, -0.339)^T$. Hence, if at the initial time the robots are in formation and $\beta(0) = \bar{\beta}$, the positions of the followers are fixed in the leader's reference frame [that is $\beta(t) = \bar{\beta}, \forall t \geq 0$, see Fig. 9(b)] and they move along concentric circles [see Fig. 9(a)].

Let us now suppose that $\omega_0(t) = 0.25 \sin(t/4)$, then $\omega_0^-(t) = -0.25 \leq \omega_0(t) \leq \omega_0^+(t) = 0.25, \forall t \geq 0$. By means of (45) we see that $\beta_1^\pm = \pm 0.177, \beta_2^\pm = \pm 0.277, \beta_3^\pm = \pm 0.349$;

Fig. 10. *Second simulation*: (a) Trajectory of the four robots. (b) Time history of β_i (solid) and bounds $\beta_i^\pm$ (dash).

Fig. 11. *Third simulation, stabilization*: (a) Trajectory of the four robots. (b) Time history of β_i . (c) Norm of F_i .

therefore, since $v_i^- \geq 0$, $i = 1, 2, 3$, by theorem 3 we have that the following invariance property holds, $\forall t \geq 0$, $\forall i = 1, 2, 3$,

$$\text{if } \beta_i^- \leq \beta_i(0) \leq \beta_i^+ \text{ then } \beta_i^- \leq \beta_i(t) \leq \beta_i^+.$$

This is shown in Fig. 10(a) and (b).

Finally, let us suppose again that $\omega_0(t) = \bar{\omega} = 0.25$ but that the robots are not initially in formation. Fig. 11(a) and (c) shows the performance of the stabilizing controller (13), with control gain $k = 1$. The robots are asymptotically in F -formation. Moreover as we can see in Fig. 11(b), $\lim_{t \rightarrow +\infty} \beta(t) = \bar{\beta}$. This is because $\bar{\beta}$ is an equilibrium point which is asymptotically stable (the robots asymptotically move along a circle): this property can be easily verified from (41).

V. CONCLUSION

In this paper, we have studied a class of hierarchical formations of unicycle robots. One of the robots plays the role of the leader and the formation is induced through a constraint function F , which depends on the position and the orientation of the robots. We have investigated the dynamics of the formation with

respect to the leader's reference frame by introducing the concept of *reduced internal dynamics* of the formation. We have characterized its equilibria and given sufficient conditions for their existence (theorems 1 and 2). The proposed theoretical results have been applied to the case in which the constraint F induces a formation where the i th robot follows a convex combination of the positions of the preceding $i-1$ robots. In this setting, we have proved an invariance property (theorem 3) which states that, if the curvature of the leader's trajectory is sufficiently small, the positions and orientations of the robots, relative to the leader's reference frame, are confined in a precise polyhedral region.

APPENDIX

A. Proof of Theorem 1

First part of the proof: 1) $\Rightarrow$ 2): Suppose that a) holds. Then $\theta_0(t) = \bar{\theta}_0$, $\omega_0(t) = 0$, $\forall t \geq 0$ and $\theta_i(t) \in \{\bar{\theta}_0, \bar{\theta}_0 + \pi\}$, $\forall i = 1, \dots, n$. First, we show that, $\forall i = 1, \dots, n$:

$$v_i = \begin{cases} 1 & \text{if } \theta_i(t) = \bar{\theta}_0 \\ -1 & \text{if } \theta_i(t) = \bar{\theta}_0 + \pi. \end{cases} \quad (54)$$

In fact, since $\mathcal{R}$ is in F -formation:

$$F(r_0(t), \dots, r_n(t)) = 0, \forall t \geq 0 \quad (55)$$

and, since F is rototranslation invariant, $\forall t \geq 0, \forall h \in \mathbb{R}$:

$$h \rightsquigarrow F\left(r_0(t) + \begin{pmatrix} \tau(\theta_0) \\ 0 \end{pmatrix} h, \dots, r_n(t) + \begin{pmatrix} \tau(\theta_0) \\ 0 \end{pmatrix} h\right) = 0.$$

Then, differentiating with respect to h , we get for $h = 0$

$$\sum_{i=1}^n \partial_{\xi_i} F(\mathcal{R}(t)) \begin{pmatrix} \tau(\theta_0) \\ 0 \end{pmatrix} = -\partial_{\xi_0} F(\mathcal{R}(t)) \begin{pmatrix} \tau(\theta_0) \\ 0 \end{pmatrix}.$$

Differentiating (55) with respect to t we have also

$$\sum_{i=1}^n \partial_{\xi_i} F(\mathcal{R}(t)) \begin{pmatrix} v_i \tau(\theta_i) \\ 0 \end{pmatrix} = -\partial_{\xi_0} F(\mathcal{R}(t)) \begin{pmatrix} \tau(\theta_0) \\ 0 \end{pmatrix}$$

since $v_0 = 1$ by assumption 1 and $r_i(t) = \begin{pmatrix} v_i \tau(\theta_i) \\ 0 \end{pmatrix}$, being $\omega_i = 0, \forall i = 1, \dots, n$. Because F is regular, v_i must satisfy (54). To prove that $\llbracket \mathcal{R} \rrbracket$ is constant, by 1) of proposition 3, it is sufficient to observe that

$$\begin{aligned} & \frac{d}{dt} \left(R(-\theta_0(t)) \left(\begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} - \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} \right) \right) \\ &= \omega_0 \left(R \left(-\theta_0(t) + \frac{\pi}{2} \right) \left(\begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} - \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} \right) \right) \\ &+ R(-\theta_0(t)) (v_i \tau(\theta_i(t)) - \tau(\theta_0(t))) = 0 \end{aligned}$$

since $\omega_0 = 0, \theta_i(t) \in \{\bar{\theta}_0, \bar{\theta}_0 + \pi\}, \forall i = 1, \dots, n$ and (54) holds.

Let us now suppose that (b) holds, then ω_0 is a nonzero constant. By (29) it follows that

$$\begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \rho_i(t) \eta(\theta_i(t)), \forall t \geq 0.$$

First of all we want to show that

$$\rho_i(t) \text{ is constant, } \forall t \geq 0 \quad (56)$$

and

$$\omega_i(t) = \omega_0, \forall t \geq 0. \quad (57)$$

In fact, (56) holds, since

$$\begin{aligned} \frac{d}{dt} (\rho_i^2(t)) &= \frac{d}{dt} \left(\left\| \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right\|^2 \right) \\ &= 2 \left\langle \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \tau(\theta_i(t)) \right\rangle = 0 \end{aligned}$$

by (29). To show (57), set $\forall s \in \mathbb{R}$

$$p(s) = \left(R(-s\omega_0) \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, s\omega_0 \right).$$

Since F is rototranslation invariant and $\mathcal{R}$ is in F -formation, it follows that, $\forall t \geq 0, \forall s \in \mathbb{R}$:

$$s \rightsquigarrow F(H_{p(s)}(\mathcal{R}(t))) = 0. \quad (58)$$

Note that $\forall (x, y, \theta) \in \mathbb{R}^2 \times S^1$ [see (16)]

$$h_{p(s)}(x, y, \theta) = \left(R(s\omega_0) \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right) + \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \theta + s\omega_0 \right)$$

and

$$\begin{aligned} & \left(\frac{d}{ds} h_{p(s)}(x, y, \theta) \right) \Big|_{s=0} \\ &= \left(\omega_0 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right) \right) \\ &= \left(\omega_0 \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix}^\perp \right) \quad (59) \end{aligned}$$

where $\begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix}^\perp$ denotes the orthogonal of $\begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix}$. Then, by taking the derivative of (58) with respect to s , computed for $s = 0$, we obtain

$$\sum_{i=1}^n \partial_{\xi_i} F(\mathcal{R}(t)) \begin{pmatrix} \omega_0 \rho_i \tau(\theta_i(t)) \\ \omega_0 \end{pmatrix} = -\partial_{\xi_0} F(\mathcal{R}(t)) \begin{pmatrix} \tau(\theta_0) \\ \omega_0 \end{pmatrix}. \quad (60)$$

Since $\mathcal{R}$ is in F -formation, $F(\mathcal{R}(t)) = 0, \forall t \geq 0$. Differentiating with respect to t , ($v_0 = 1$), we get $\forall t \geq 0$

$$\sum_{i=1}^n \partial_{\xi_i} F(\mathcal{R}(t)) \begin{pmatrix} v_i \tau(\theta_i(t)) \\ \omega_i(t) \end{pmatrix} = -\partial_{\xi_0} F(\mathcal{R}(t)) \begin{pmatrix} \tau(\theta_0(t)) \\ \omega_0 \end{pmatrix}. \quad (61)$$

Since F is regular, by (60) and (61) it follows that $\omega_i(t) = \omega_0, v_i(t) = \rho_i \omega_0, \forall t \geq 0$, then (57) holds, which implies in particular that there exist $\bar{\theta}_1, \dots, \bar{\theta}_n \in S^1$ such that $\forall i = 1, \dots, n$

$$\theta_i(t) - \theta_0(t) = \bar{\theta}_i, \forall t \geq 0. \quad (62)$$

Now we obtain that $\forall i = 1, \dots, n$

$$\begin{aligned} R(-\theta_0(t)) \left(\begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right) &= \rho_i \begin{pmatrix} \tau^T(\theta_0) \\ \eta^T(\theta_0) \end{pmatrix} \eta(\theta_i) \\ &= \rho_i \begin{pmatrix} \sin(\theta_0 - \theta_i) \\ \cos(\theta_0 - \theta_i) \end{pmatrix} \end{aligned}$$

which is constant by (56) and (62). Then $\forall i = 1, \dots, n, \forall t \geq 0$:

$$\begin{aligned} & R(-\theta_0(t)) \left(\begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} - \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} \right) \\ &= R(-\theta_0(t)) \begin{pmatrix} x_i(t) - \bar{x} \\ y_i(t) - \bar{y} \end{pmatrix} - R(-\theta_0(t)) \begin{pmatrix} x_0(t) - \bar{x} \\ y_0(t) - \bar{y} \end{pmatrix} \\ &= \bar{q}_i \in \mathbb{R}^2 \end{aligned}$$

which implies that $\forall t \geq 0$

$$H_{-r_0(t)}(\mathcal{R}(t)) = ((0, 0), (\bar{q}_1, \bar{\theta}_1), \dots, (\bar{q}_n, \bar{\theta}_n))^T = \bar{\xi}$$

then 1) holds by remark 5.

Second Part of the Proof: 2) plus condition (30) $\Rightarrow$ 1): By 1) of proposition 3 and definition 6, there exist $\bar{\theta} = (\bar{\theta}_0, \dots, \bar{\theta}_n)^T, \bar{P}_1, \dots, \bar{P}_n \in \mathbb{R}^2$ such that, $\forall i = 1, \dots, n, \forall t \geq 0$

$$\theta_i(t) - \theta_0(t) = \bar{\theta}_i, \quad (63)$$

$$R(-\theta_0(t)) (P_i(t) - P_0(t)) = \bar{P}_i \quad (64)$$

where $P_i(t) = \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix}, \forall i = 1, \dots, n$.

Differentiating (64), recalling that $v_0 = 1$, we obtain that

$$R(-\theta_0)(v_i\tau(\theta_i) - \tau(\theta_0)) - \omega_0 R\left(-\theta_0 + \frac{\pi}{2}\right)(P_i - P_0) = 0$$

that is

$$v_i\tau(\theta_i - \theta_0) - \tau(0) - \omega_0 R\left(-\theta_0 + \frac{\pi}{2}\right)(P_i - P_0) = 0.$$

By left multiplying this expression by $R(\theta_0 - \theta_i)$ it follows that $\forall i = 1, \dots, n$:

$$v_i\tau(0) - \tau(\theta_0 - \theta_i) - \omega_0 R\left(-\theta_i + \frac{\pi}{2}\right)(P_i - P_0) = 0$$

whose second component is

$$\sin(\theta_i - \theta_0) - \omega_0 \langle \tau(\theta_i), P_i - P_0 \rangle = 0. \quad (65)$$

Note that

$$\langle \tau(\theta_i), P_i - P_0 \rangle \neq 0, \forall i = 1, \dots, n. \quad (66)$$

In fact, if this were not true, it should be (for some t)

$$\langle \tau(\theta_i), P_i - P_0 \rangle = 0 \text{ and } \theta_i \in \{\theta_0, \theta_0 + \pi\}$$

since $\sin(\theta_i - \theta_0) = 0$ by (65) which is impossible by hypothesis (30). Moreover,

$$\langle \tau(\theta_i), P_i - P_0 \rangle \text{ is constant} \quad (67)$$

by (63), (64), and because it is the second component of $R(\theta_0 - \theta_i)R(-\theta_0)(P_i - P_0)$. Therefore, by (63), (65), (66), and (67) we have that ω_0 is constant. Since $\sin(\theta_i - \theta_0) = \langle \tau(\theta_i), \eta(\theta_0) \rangle$, we have, by (65)

$$\langle \tau(\theta_i), \eta(\theta_0) + \omega_0(P_i - P_0) \rangle = 0. \quad (68)$$

Then, if $\omega_0 \neq 0$ it follows that $\langle \tau(\theta_i), P_i - (P_0 + (1/\omega_0)\eta(\theta_0)) \rangle = 0$. Since $P_0 + (1/\omega_0)\eta(\theta_0)$ is constant, being $(d/dt)(P_0 + (1/\omega_0)\eta(\theta_0)) = 0$, 2) is verified setting $(\frac{x}{y}) = P_0 + (1/\omega_0)\eta(\theta_0)$. If $\omega_0 = 0$, we obtain 1) by (68). ■

Proof of Lemma 1: Recall that $\Theta : \Gamma_F \rightarrow T^n$, is defined by $\Theta(\xi) = (\theta_1, \dots, \theta_n)^T$, which is the projection of Γ_F on the torus (where $\xi = (0, \xi_1, \dots, \xi_n) \in \Gamma_F$ and $\xi_i = (x_i, y_i, \theta_i)^T$, $\forall i = 1, \dots, n$). Note that Θ satisfies the following properties:

- 1) Θ is proper, that is $\Theta^{-1}(K)$ is compact for any compact $K \subset T^n$. In fact if K is compact in T^n , $\Theta^{-1}(K)$ is closed by the continuity of Θ (since Θ is a projection); hence, $\Theta^{-1}(K)$ is compact since $\Theta^{-1}(K)$ is bounded being a subset of Γ_F which is contained in $(B((0), \rho_F)) \times S^1)^n$, (note that $\rho_F < +\infty$ by hypothesis).
- 2) Θ is a local diffeomorphism by (31) and the implicit function theorem. Therefore, the couple (Γ_F, Θ) is a covering map of T^n by proposition 2.19 of [42], which implies that Θ is surjective (therefore 1) holds) and has the unique path lifting property by proposition 6.7 of [43]. Since $\mathcal{F}_r$ is not empty by (2), Γ_F is not empty. Set $D =]-\pi/3, \pi/3[^n \subset T^n$ and consider $\Theta^{-1}(D)$ which is non-empty since Θ is surjective being a covering map. Let $\tilde{\Theta}^{-1}(D)$ be a connected component of $\Theta^{-1}(D)$. Let $\tilde{\Theta} : \tilde{\Theta}^{-1}(D) \rightarrow D$ be the restriction of Θ to $\tilde{\Theta}^{-1}(D)$, then $\tilde{\Theta}$ has the unique path

lifting property (see for instance [43, Sec. 6.4]). Therefore, $\tilde{\Theta}$ is a (global) diffeomorphism by Corollary 6.9 of [43].

Suppose now that $\bar{\omega} \neq 0$, set any $i : 1 \leq i \leq n$ and observe that for any choice of $\bar{\theta}_1, \dots, \bar{\theta}_{i-1} \in [-\pi/2, \pi/2]$, the map

$$\alpha_{\bar{\theta}_1, \dots, \bar{\theta}_{i-1}} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

defined by $\alpha_{\bar{\theta}_1, \dots, \bar{\theta}_{i-1}}(\theta_i) = \arg(\Pi_i(\tilde{\Theta}^{-1}(\bar{\theta}_1, \dots, \bar{\theta}_{i-1}, \theta_i, 0, \dots, 0)) - \begin{pmatrix} 0 \\ 1/\bar{\omega} \end{pmatrix}) + \pi/2$, is continuous, therefore it has a fixed point $\bar{\theta}_i$ that is

$$\bar{\theta}_i = \arg\left(\Pi_i(\tilde{\Theta}^{-1}(\bar{\theta}_1, \dots, \bar{\theta}_{i-1}, \bar{\theta}_i, 0, \dots, 0)) - \begin{pmatrix} 0 \\ 1/\bar{\omega} \end{pmatrix}\right) + \frac{\pi}{2}. \quad (69)$$

Therefore, iterating on $i = 1, \dots, n$, we can find $\bar{\theta} = (\bar{\theta}_1, \dots, \bar{\theta}_n)^T \in [-\pi/2, \pi/2]^n$ such that (69) is verified for any $i = 1, \dots, n$. Note that $\forall (\theta_1, \dots, \theta_n)^T \in [-\pi/2, \pi/2]^n$

$$\partial_{\theta_j}(\Pi_i(\tilde{\Theta}^{-1}(\theta_1, \dots, \theta_n))) = 0, \forall i = 1, \dots, n-1, \forall j = i, \dots, n$$

by (3) and the implicit function theorem. Therefore, $\Pi_i(\tilde{\Theta}^{-1}(\bar{\theta}_1, \dots, \bar{\theta}_i, 0, \dots, 0)) = \Pi_i(\tilde{\Theta}^{-1}(\bar{\theta}_1, \dots, \bar{\theta}_n))$, $\forall i = 1, \dots, n$ which implies by (69) that

$$\bar{\theta}_i = \arg\left(\Pi_i(\tilde{\Theta}^{-1}(\bar{\theta}_1, \dots, \bar{\theta}_n)) - \begin{pmatrix} 0 \\ 1/\bar{\omega} \end{pmatrix}\right) + \frac{\pi}{2}, \forall i = 1, \dots, n.$$

Then $\bar{\xi} = \tilde{\Theta}^{-1}(\bar{\theta}_1, \dots, \bar{\theta}_n)$ verifies (35) since $\bar{\theta}_i = \Theta_i(\bar{\xi})$, $\forall i = 1, \dots, n$. Hence, 2) holds. ■

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